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Entry under placement uncertainty

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Keywords

Placement uncertainty, over entry, under entry, cap-and-trade regulations

Disciplines

Environmental Policy | Industrial Organization

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1 Introduction

The paper studies the role of *placement* or *rank uncertainty* - a firm's uncertainty about its productive efficiency relative to its rivals' - on entry decisions and industry performance, within the context of a cap-and-trade (CAT) regulation. CAT is used worldwide to manage negative production externalities in environmental and natural resource-based industries and other congested settings.[1] The regulation places an upper bound on either the aggregate by-product of a production process, e.g., in the case of pollution emissions, or on the aggregate industry output directly, e.g., in the case of a fishery. Firms operating under these regulations must hold shares of the aggregate cap in quantities that match own production. When the cap binds in the aggregate, the permits held by one firm displace, one-for-one, the permits available to competing producers. Thus, remaining active requires individual firms to match the productivity of their competitors; if they cannot, they are better off selling their permits to more productive rivals and exiting the industry.

A CAT-regulated industry provides a natural and analytically tractable setting to study the effect of placement or rank uncertainty on a firm's entry decisions. Firms in our model are heterogeneous in cost efficiency. A firm knows its own costs but is uncertain about the mean cost efficiency among a population of potential entrants/competitors and consequently about its productivity rank. Average efficiency determines post-entry permit trading prices and returns to the firm's capital investment. It must therefore base its entry decision on a subjective prior of the competition it will face post entry. The specific question we ask is, under what information structures and beliefs about competitor productivity will there be excess or insufficient entry compared to a full information benchmark, in which mean cost efficiency and consequently firms' efficiency ranking in an entrant population is known?

Our main and we believe novel result, is that *uncertainty* over one's productivity rank is sufficient to cause inefficient over-entry under specified conditions and inefficient under-entry under contrasting conditions, relative to a full information benchmark. Equilibrium permit prices and returns to vested capital are determined by the number and productivity of *entering* firms, specifically, by a firm's cost efficiency relative to that of its rival entrants. We show that when firms have Bayesian beliefs on the cost distribution of the population, while believing themselves to be the median cost firm, the entry may be excessive and industry wide efficiency lower than under full information − when the true population average cost is low. Vice versa, for higher true average population cost placement uncertainty may cause insufficient entry relative to full information.

Our paper contributes to a behavioral IO literature on market entry in the presence of "competitive blind spots." This literature often attributes excess market entry to overconfidence, i.e., entrepreneurs who underestimate the competition they will face post entry and therefore overestimate return on investment (Rapoport et al. (1998); Camerer and Lovallo (1999); Cain et al. (2015); Koellinger et al. (2007); Schüssler (2018)). The biased assessment of potential competition leads to a sub-optimal decision to enter an industry or invest in a venture. Our work shows that such outcomes exist in a rational expectations market equilibrium. In

[1] The European Union Emissions Trading System regulates carbon dioxide emissions from the 28 European Union countries, plus Iceland, Liechtenstein, and Norway. See Narassimhan et al. (2018) for a recent review of cap-and-trade pollution regulations. Marine commercial fisheries have been regulated with tradable quotas since 1983 (in New Zealand). Currently 16 major U.S. fisheries are managed with tradable fishing quota regulations. A full list of world, CAT-regulated fisheries is available in Lynham (2014).

particular, we find that a firm who views itself as *average* but faces uncertainty over its productivity rank among its competitors will make similarly sub-optimal investment decisions.[2]

Although the framework adopted in the paper is that of a CAT-regulated industry, our results apply more generally. An endogenously determined permit price can be replaced by an endogenously determined goods price obtained under an inelastic product demand. Firms in this setting must compete for customers; the relative productivity of competing firms continues to play a central role in the returns to capital and entry. Thus, market forces that drive our results are not specific to CAT regulations. The unique feature of a CAT-regulated industry pointed out earlier sharpens the role of the relative productivity argument and simplifies the analysis by shutting out multiple general equilibrium effects.

The baseline model is a standard global game with private values[3] but one in which players' actions are strategic substitutes rather than complements.[4] We show the existence of an unique equilibrium in a set up with endogenous, albeit parameterized payoffs. To this extent, our results contribute to the rather sparse literature on global games with strategic substitutability (e.g. Karp et al. (2007); Harrison and Jara-Moroni (2015); Morris and Shin (2009)).

Our work also contributes to the recent and fast growing literature on welfare analysis in economies with incomplete private information. This literature highlights the dual nature of prices - as conveyors of information and as determinants of resource allocation. Moreover, various types of inefficiencies - aggregative or distributional - are traced to externalities arising from this dual role (see e.g. Morris and Shin (2002, 2005); Angeletos and Pavan (2007, 2009); Amador and Weill (2010); Vives (1993, 1997, 2017)). The literature primarily focuses on the relative values of private and public information (as conveyed by prices) in a framework of ex-ante (but not ex-post, since they receive different signals) identical agents facing common shocks. The set of agents participating in the market is assumed to be constant - all agents are active. The focus of our paper is the agent's decision itself of whether or not to enter and compete for a fixed production quota. The set of active firms is thus endogenous. Within the context of this decision problem, we identify and model a new source of strategic uncertainty that is associated with agent heterogeneity - namely, placement uncertainty.

The literature on firm dynamics under complete and incomplete information is vast. Jovanovic (1982) studies firm dynamics in a setting where firms are uncertain and learn their own productivity after entry has occurred. We reverse the information set up; we envision an industry in which firms are aware of their own productivity but are unsure about others'. The CAT regulation, by pricing a production externality, forces firms to carefully assess their relative productivity and determine whether or not they can profitably secure shares of the total production quota. Our model can therefore be interpreted as one of strategic exit.

Ghemawat and Nalebuff (1985, 1990) study exit in an exogenously declining industry. The question is, which of the incumbents will exit or reduce their capacity first. The authors find that larger firms either exit, in the all or nothing version of the model (Ghemawat and Nalebuff, 1985), or reduces productive capacity first, in a more general version (Ghemawat and Nalebuff, 1990). Fudenberg and Tirole (1986) introduce

[2]Similarity between overconfident entry and entry under placement uncertainty is investigated in an extended appendix.

[3]Payoffs are a function of the private signals received by the players, their actions and the actions of the rival players.

[4]The literature on global games with strategic complementarities is substantial and include the well known works of Carlsson and Van Damme (1993); Morris and Shin (1998, 2001, 2005); Frankel and Pauzner (2000); Frankel et al. (2003) among others.

incomplete information into a dynamic duopoly competition game. As in our model, firms in Fudenberg and Tirole (1986) know their own productivity (costs) but must choose when if ever to exit based on their beliefs about rival costs. Over time, active firms become more pessimistic about the cost of their rival. In long run equilibrium, an industry shake out occurs with high cost firms exiting and low cost firms remaining active.

Our paper offers new insights for understanding the effects of environmental regulations on industry structure and performance (see reviews in Heyes (2009) and Millimet et al. (2009)). Much of this literature has focused on taxation, and command-and-control type regulations. The determinants of, and the role that production permit prices play in industry structure has not been fully vetted. Finally, our analysis of placement uncertainty on entry decisions and industry performance contributes to the behavioral industrial organization literature that emphasizes the market implications of bounded rationality of firms and consumers (Tremblay et al. (2018); see also Hoffman and Burks (2017)).

The paper is organized as follows. The next section introduces placement uncertainty into a simple two-firm game of non-cooperative entry. We derive, in the simplest setting, the post entry quota price, returns to capital and quota, and entry behavior. The insights are then extended and refined for the case of a continuum of firms. Section 3 presents the continuum firm model. Section 4 derives the equilibrium permit price and entry under a benchmark full information scenario. Section 5 presents the results under incomplete information. Section 6 compares industry structure and performance under contrasting information structures. Section 7 summarizes our results and discusses some implications for the behavioral industrial organization literature. Proposition proofs are collected in an appendix.

2 Two firm model

We consider a single production period during which an industry is regulated to produce no more than Q units of output. We assume Q has been pre-determined by a planner to meet a broader social goal.[5] The regulation requires that units of production be matched with units of quota. Quota has no value outside the industry or beyond the single production period under consideration. There is no cheating in our model, i.e., no over quota production. Individual firm production and quota is denoted in common units, q. We will refer to q as quota units synonymously as permits. Industry output is sold at constant unit price, p.

Firms decide simultaneously whether to enter the quota-regulated industry. Entry implies commitment of a single unit of capital for the full production period at irreversible cost $\delta > 0$, which we assume is common across firms.

Firms are heterogeneous in own productivity, which manifests as differences in variable costs of production. We use θ_i to denote an inverse productivity measure for firm $i = 1, 2$ with larger values corresponding to higher costs, i.e., lower efficiency. We use $c(q|\theta_i)$ to denote the variable cost given θ_i. Variable costs are assumed strictly convex in q, with $c(0|\cdot) = 0$. Our analysis will feature values of $\theta_i > 0$, i.e, costs that are strictly increasing in production, although this property is not essential for our results. The structural property

[5] Our model determines capital entry and thus the total cost of producing a given Q under varying information structures. We do not consider the problem of setting the *optimal* quota.

we require is eventual decreasing returns to scale.

The following functional form satisfies these properties and allows for sharp analytical results:

$$c(q|\theta_i) = \theta_i q + \frac{1}{2}\lambda q^2, \tag{1}$$

where $\lambda > 0$ is common across firms. The cost functional form of (1) will be used throughout the analysis.

Without loss of generality, we designate firm 1 as the more efficient of the two firms; $\theta_1 < \theta_2$. For reasons that will become clear subsequently, we fix the cost parameter difference between the two firms at $\theta_2 - \theta_1 = 2\varepsilon$. We use $\theta \equiv \frac{\theta_1+\theta_2}{2}$ to denote the average of the two cost parameters. Implicitly, $\{\theta_1, \theta_2\} = \{\theta - \varepsilon, \theta + \varepsilon\}$.

We assume that Q has been allocated to the two firms. Both firms participate in a post-entry permit trading market. Firms announce a net trade schedule to a market maker who aggregates schedules and determines a market clearing quota price and the purchases/sales for each firm. All firms are required to transact according to their reported net trade schedule (Malueg and Yates, 2009).

2.1 Quota market equilibrium and firms' entry choice

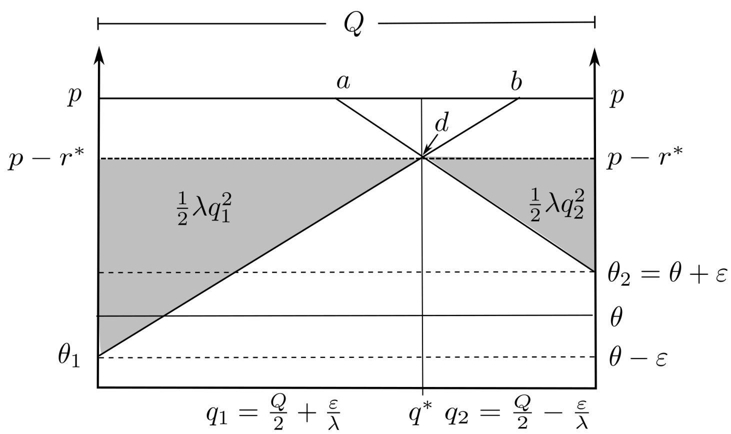


Figure 1: **Quota market and rents.**

Figure 1 illustrates the quota market under a representative parameterization of the model. Moving left to right on the horizontal axis denotes the quota and production for firm 1; moving right to left denotes quota and production for firm 2. The length of the horizontal axis is Q. Units on the vertical axis are in dollars. Maximum industry revenue is pQ.

Diagonal lines denote marginal costs with intercept θ_i for firm i at $q_i = 0$; the slope of the marginal cost curves is λ (measured relative to positive production). The two-firm average, θ, is indicated on the vertical axis.

Suppose first that only firm 1 has entered. Profit maximizing production equates price p and marginal cost at b. In this case the aggregate quota constraint is slack with $q_1 < Q$. Firm 1's variable profit is triangle $\theta_1 pb$. Thus entry by firm 1 is optimal if and only if area $\theta_1 pb \geq \delta$.

When industry output is less than Q, the equilibrium quota trading price, which we denote r^*, will be equal to zero. Moreover when the quota is slack, industry profit flows entirely to the physical capital that is allocated to the industry.

Now suppose both firms have entered. If both firms produce the quantities that equate price and marginal cost, aggregate production will exceed Q. Thus, Q must be rationed. The efficient quota allocation, conditional on the set of entrants (firms 1 and 2 in this case) occurs at q^* where the marginal profit and marginal cost, $\theta_i + \lambda q_i$, is equal for each entrant.

When the aggregate quota binds total industry profit flows to two fixed factors: $r^* Q$ is the rent allotted to the fixed quota; the remaining rent is allotted to the physical capital that has been committed. In figure 1, firm 1's capital rent is the triangle $\theta_1(p - r^*)d$; firm 2's capital rent is $\theta_2(p - r^*)d$. Both firms will cover capital costs and enter if and only if capital rent is greater than δ.

Note that while the quota binds, the quantities $\{q_1, q_2\}$ and the respective capital rent is invariant to the average cost parameter θ. This property is easily seen in figure 1 where it is apparent that the capital rent triangles will shift vertically with θ while as long as the quota constraint binds, individual firm production and capital rents are unchanged.

When the quota binds, the return to capital for the high cost firm is: $\frac{1}{2\lambda}\left(\frac{\lambda Q}{2} - \varepsilon\right)^2 \equiv \underline{\delta}$. Note that $\underline{\delta}$ presumes that θ be sufficiently low such that the marginal cost at point d not exceed p. It is clear from figure 1 that both firms will enter the industry if and only if $\delta \leq \underline{\delta}$.

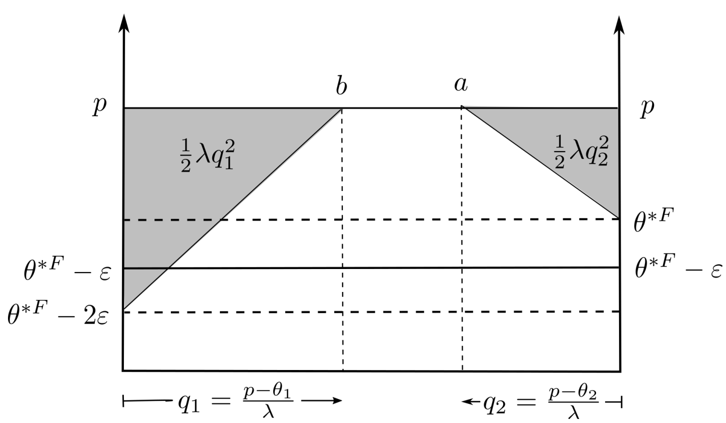


Figure 2: **Slack quota constraint.**

Figure 2 makes clear that the quota constraint will be slack for sufficiently high values of θ. In this case, firms' production quantities and capital rents are decreasing in θ; q_i and $\frac{1}{2}\lambda q_i^2$ both decline as θ increases.

Assume next that firm 2's capital rent $\theta^{*F} pa$ in figure 2 is exactly equal to δ (superscript F distinguishes the full information scenario). If $\theta_1 = \theta^{*F}$, firm 1 would earn the same capital rent. Thus, only one firm will enter the industry if $\theta \in (\theta^{*F} - \varepsilon, \theta^{*F} + \varepsilon]$ and no firms will enter when $\theta > \theta^{*F} + \varepsilon$.

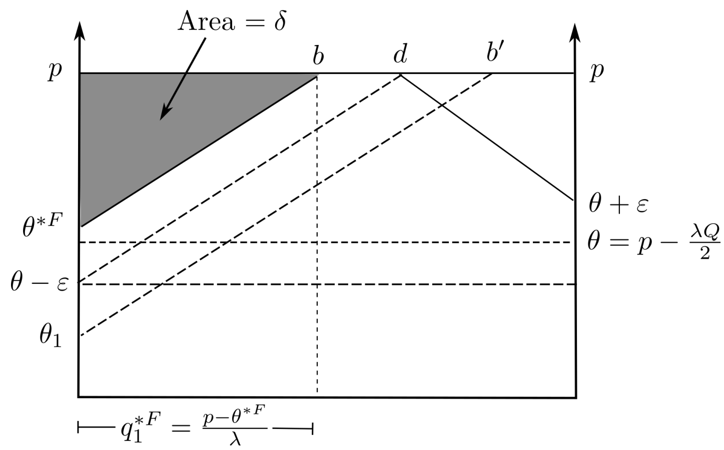


Figure 3: **Firm heterogeneity, rent outcomes, and entry.**

Figure 3 shows the case where $\theta = p - \frac{\lambda Q}{2}$ with $r^* = 0$. In this example, marginal costs of both firms evaluated at quantities delineated by the intersection point d are equal to p, with Q fully utilized when both firms enter the market.

Now suppose $\delta > \underline{\delta}$. Firm 2 never enters since its variable profit is $\underline{\delta} < \delta$. Firm 1 enters and earns capital rent given its lower value of θ_1, for example, area $\theta_1 p b' > \delta$. There will exist a higher value, $\theta = \theta^{*F} + \varepsilon$, where firm 1's capital rent triangle $\theta^{*F} p b = \delta$. We conclude that no entry occurs if $\theta > \theta^{*F} + \varepsilon$. The figure makes clear that the larger is the value of δ, the lower will be the cutoff θ^{*F}. Not surprisingly, when capital costs are high, variable costs must be low in order to induce entry into the industry.

Several insights emerge from this simple model.

First, if the quota constraint binds, an increase in p increases quota rent $r^* Q$, while leaving capital rent unchanged. This is not the case when the quota constraint is slack. Second, the number of firms that enter depends on (i) the technology, (ii) the regulation, and if and only if the quota constraint is slack, (iii) the market demand. These three elements are reflected in our model, respectively, as cost parameters $(\theta_i, \lambda, \delta)$, quota Q, and output price p.

Finally, the two-firm example shows that entry decisions are strategically independent when the quota constraint is slack. By contrast, when the quota constraint binds, capital allocations are strategic substitutes. Capital rent depends on the number and the average productivity of entering firms. Payoff interaction operates through the equilibrium quota price r^* which distributes industry variable profit to the fixed quota and to vested capital.

2.2 Equilibrium entry under placement uncertainty

Our objective now is to understand entry decisions when firms do not know where they rank *vis a vis* their competitors in terms of production costs. The informational environment is as follows. First, nature chooses

$\theta \sim U\left[\underline{\theta}, \bar{\theta}\right]$, where $\underline{\theta}$ and $\bar{\theta}$ are set to economically meaningful values (see appendix 8.1). Then the two firms are randomly assigned $\theta_i \in \{\theta_1, \theta_2\} = \{\theta - \varepsilon, \theta + \varepsilon\}$. θ_i is firm i's private information. A firm i forms a Bayesian posterior by assigning equal probability to its rival's true cost as $\theta_{j \neq i} \in \{\theta_i - 2\varepsilon, \theta_i + 2\varepsilon\}$.[6] We term this environment as *placement* uncertainty.

The question is: How do entry decisions under placement uncertainty compare with full information?

The analysis above shows that the answer to this question must consider differing cost scenarios. First, consider the entry choice when $\delta \leq \underline{\delta}$. A firm earns a capital rent of $\underline{\delta}$ when both firms enter and the quota constraint binds. The highest θ_i for which a firm can still earn $\underline{\delta}$ occurs when the quota binds with $r^* = 0$ is $\theta_i = p - \frac{\lambda Q}{2} + \varepsilon$. A firm with θ_i above this value has no market interaction with its rival as is the case under full information. Its strategy is therefore unchanged: Enter if and only if $\theta_i \leq \theta^{*F}$. In a low capital cost scenario, incomplete information does not alter the entry choice. Below we focus on the more interesting case when $\delta > \underline{\delta}$. We show that placement uncertainty then plays a critical role in firms' entry decisions.

We seek a pure strategy Bayesian equilibrium in which firms' entry decision depends on a threshold value for its cost parameter: firm i enters if and only if $\theta_i \leq \theta^{*I}$, where the superscript I is added to distinguish entry threshold values under incomplete information.

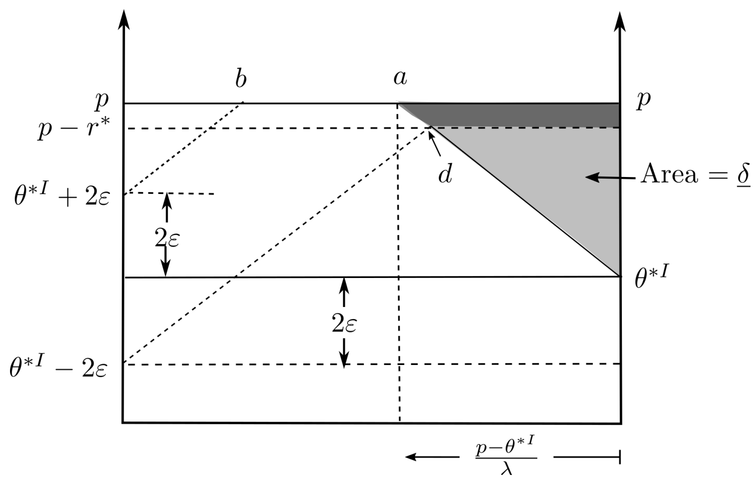


Figure 4: **Entry under placement uncertainty.**

Figure 4 illustrates how the threshold θ^{*I} is determined. The figure shows firm i's production originating from the right side. Firm i is assumed to be the pivotal firm with $\theta_i = \theta^{*I}$. This firm is pivotal in the sense that it expects to earn capital rent just equal to δ. Firm i assigns equal probability that its rival has cost parameter θ_j higher or lower than is own by the amount 2ε. These possibilities are denoted on the left vertical axis in figure 4. Firm i believes that if j has the low cost $\theta^{*I} - 2\varepsilon$, it will enter, in which case the quota will bind and i will earn capital rent $\underline{\delta}$. On the other hand, if j is high cost, $\theta_j = \theta^{*I} + 2\varepsilon$, firm i will be the

[6] These Bayesian posteriors apply to the domain of θ realizations that are relevant for the strategic decision of the firms. It is easily checked that the decision rules remain unaltered when θ realizations are close to the bounds.

only entrant. The quota constraint will then be slack, and firm i will earn capital rent, area $\theta^{*I}pa$. Thus, i's expected capital rent is the average of $\underline{\delta}$ and $\theta^{*I}pa$. This expectation equals $\delta > \underline{\delta}$ precisely at $\theta_i = \theta^{*I}$. It is clear from figure 4 that a firm with $\theta_i > \theta^{*I}$ will choose not to enter.

It remains to be established that firm i will always enter if $\theta_i < \theta^{*I}$. To see this, consider $\theta_i < \theta^{*I} - 2\varepsilon$. Now i is convinced that its rival will also enter because even at $\theta_j = \theta_i + 2\varepsilon$, j's cost is below the threshold θ^{*I}. Equally likely is the case that firm j has a lower cost in which case both firms will enter and the quota constraint will bind. Firm i therefore expects to earn a capital rent equal to an average of the two shaded areas shown in figure 1. Firm i will enter only if this average exceeds δ.

Define $\bar{\delta} \equiv \frac{1}{2\lambda}\left(\left(\frac{\lambda Q}{2}\right)^2 + \varepsilon^2\right)$ as this expectation. Then firm i enters if and only if $\delta \leq \bar{\delta}$. Entry is thus an equilibrium choice for all $\theta_i \leq \theta^{*I}$, conditional on $\delta \in \left(\underline{\delta},\ \bar{\delta}\right)$.

It is easily seen that when $\theta < \theta^{*I} - \varepsilon$, both firms' cost parameter realizations fall below threshold θ^{*I} and thus both will enter. On the contrary, when $\theta > \theta^{*I} + \varepsilon$, both firms have $\theta_i > \theta^{*I}$ and no entry occurs. In between, i.e., $\theta \in \left(\theta^{*I} - \varepsilon, \theta^{*I} + \varepsilon\right)$, only the low cost firm enters.

We are ready to answer the question: How does entry under placement uncertainty compare with that under full information? Conditional on our assumption $\delta \in \left(\underline{\delta}, \bar{\delta}\right)$ the answer is as follows. By construction, only one firm enters under full information for $\theta_1 \leq \theta^{*F}$. Under placement uncertainty, for sufficiently low realizations of θ, both firms enter. When firms are uncertain about their competitor's cost, they base their entry decision on expected capital rent. Therefore, excess entry will occur for lower realizations of θ.

The distortion is reversed for high realizations of θ. Recall that under full information, the entry threshold θ^{*F} gets the low cost firm 1 capital rent equal to δ, i.e., the shaded area in figure 3. Now consider the lower cost firm with $\theta_1 = \theta^{*I}$ under placement uncertainty. The firm expects capital rent equal to $\theta^{*I}pa$ - the shaded area in figure 4 - with probability one half and with probability one half expects a rent $\underline{\delta} < \delta$. Therefore the shaded area $\theta^{*I}pa$ must be larger than δ for the firm with θ^{*I} under incomplete information to get an average capital rent of δ. Thus the entry thresholds under the two information structures satisfy $\theta^{*I} < \theta^{*F}$. We conclude that for values of $\theta \in \left(\theta^{*I} + \varepsilon, \theta^{*F} + \varepsilon\right)$, no entry occurs under placement uncertainty whereas the lower cost firm, under full information, enters. Thus placement uncertainty results in under entry at high realizations of θ.

These results are summarized in the following proposition and figure.

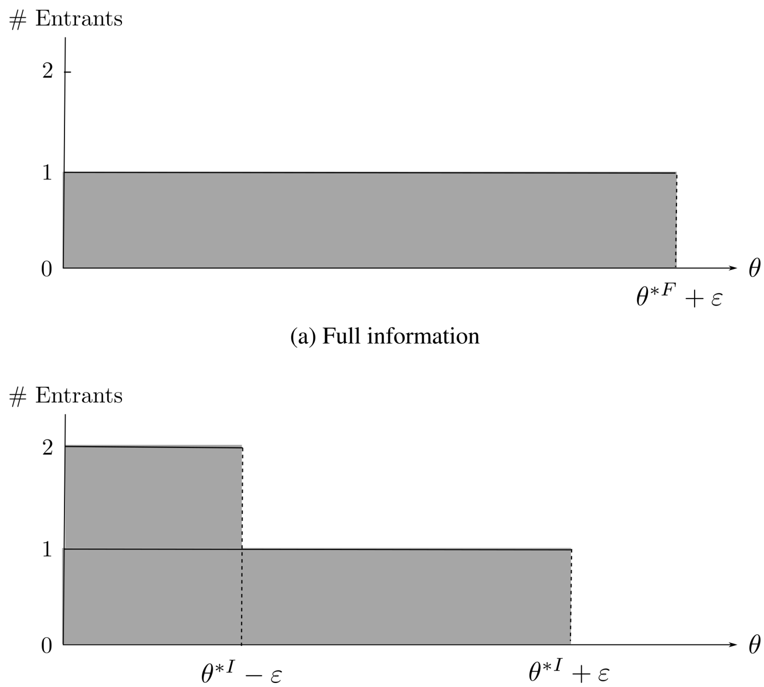
Proposition 1. *Let $\delta \in \left(\underline{\delta}, \bar{\delta}\right)$. Then with two heterogenous firms with $\{\theta_1, \theta_2\} = \{\theta - \varepsilon, \theta + \varepsilon\}$ the following entry equilibrium holds:*

(i) Under full information, only the lower cost firm enters for all $\theta \leq \theta^{*F} + \varepsilon$. No entry occurs for $\theta > \theta^{*F} + \varepsilon$.

(ii) Under placement uncertainty, both firms enter for $\theta \leq \theta^{*I} - \varepsilon$, only one firm enters for $\theta \in (\theta^{*I} - \varepsilon, \theta^{*I} + \varepsilon]$. No entry occurs for $\theta > \theta^{*I} + \varepsilon$.

(iii) Moreover,

$$\theta^{*F} > \theta^{*I}.$$



(a) Full information

(b) Incomplete Information

Figure 5: **Equilibrium entry under placement uncertainty: two firms.**

Proof: See appendix 8.1.

Figure 5 compares equilibrium entry for varying values of θ and for our two information scenarios. For each panel dashed vertical lines use used to indicate thresholds at which the number of entrants drops: in case of full information from 1 to 0 at $\theta^{*F} + \varepsilon$ and in the case of placement uncertainty from 2 to 1 at $\theta^{*I} - \varepsilon$, and from 1 to 0 at $\theta^{*I} + \varepsilon$.

To sum up, for low θ realizations the capital rent of the higher cost firm is below the cost of capital. This firm stays out under full information. Under placement uncertainty both firms expect to face quota market competition due to other's entry. However, the capital rent of the ex-post lower cost firm is sufficiently high to raise the expected (average) capital rent above the cost of capital and thus both firms enter. For high realizations of θ, capital rents are much reduced even when there is no market competition and firms capital rent depends on firm's θ_i. The firm with θ^{*F} under full information remains the only entrant. But the firm with θ^{*I} under placement uncertainty expects it will face competition in the quota market with a probability one half. The fear of competition thus depresses the entry threshold relative to full information.

This section introduced placement uncertainty into a two-firm non-cooperative entry game to get a simple understanding of the entry equilibrium under placement uncertainty. However, one drawback of this setup by construction is that a single entrant freely accesses the entire industry's quota under full information. A more pragmatic contrast of the two information scenarios calls for a setup that admits an operational quota market under both information scenarios. Another limitation of this setup is that even though industry average cost parameter θ varies continuously, the equilibrium entry is limited to $\{0, 1, 2\}$ entrants thus ruling out richer comparative statics. This is because a discrete two-firm setup lacks a continuous relationship between firms' heterogeneous costs and capital rents. A continuous dispersion of capital rents is likely to create a pivotal firm that is indifferent between entering and staying out and thus yield a finer relationship between average industry cost parameter θ and the cost characteristics of the pivotal firm.

Equally importantly, a two-firm analysis leaves out a full understanding of a key feature of the equilibrium under placement uncertainty, namely the firms' capital rent payoff as a function of θ. It is clear from the two-firm example that for low θ realizations, the quota market binds and firms' capital rent payoffs are convex in their ex-post market share. As market shares depend on heterogenous costs, placement uncertainty will lead to a higher cost of entry threshold when quota market is expected to bind. This fact is obscured in the two-firm example. In a similar vein, the payoff function is convex in market share for low probable θ realization for the θ^{*I} firm, but it falls flat for equally probable high θ realization. This payoff function truncation turns out to be crucial for an equilibrium entry threshold under placement uncertainty that is below the full information counterpart. Once again, this feature stays buried in the simplicity of a two-firm model.

The model of the next section considers entry under placement uncertainty with a continuum of heterogenous cost firms. We are then able to focus on the perceived capital rent of the pivotal entrant, highlight the role of convexity/quasi-convexity of capital rent payoffs, and thus further refine and expand on the results presented in proposition 1.

3 The model with a continuum of firms

In this section we assume there is a *continuum* of firms and normalize their mass to unity. Firms are indexed as i on this continuum and their idiosyncratic productivity θ_i follows a continuous distribution as presented below. The remaining model components are as above.

We assume that Q has been allocated *gratis* to a subset of firms with $w_i \geq 0$ denoting the allocation to firm i.[7] Let v_i denote net quota demand for firm i. If v_i is positive (negative), i is a net quota buyer (seller). Because one cannot sell more quota than is held, trade is constrained by $w_i + v_i \geq 0$.

We use $\mathcal{I}_i$ to denote the information available to firm i when the decision to enter is made. In addition to own cost efficiency, θ_i, parameters $(p, Q, \lambda, \delta, w_i) \in \mathcal{I}_i$ under all information scenarios. To reduce notation the inclusion of these five parameters is henceforth suppressed. The scenarios we contrast differ in terms of what firm i knows about the cost efficiency of other firms in the population.

We assume θ_i is distributed uniformly within the firm population around a population mean of θ and a spread of 2ε. Further, it is assumed that the distribution $\theta_i \sim U[\theta-\varepsilon, \theta+\varepsilon]$ is common knowledge. Under full information, $\mathcal{I}_i = \{\theta_i, \theta, \varepsilon\}$, each firm knows its own efficiency, the population mean value, and thus knows its efficiency *rank*. Under our *incomplete information* scenario, $\mathcal{I}_i = \{\theta_i, \varepsilon\}$. In this case, each firm knows its own efficiency and the distribution range but does not know the population mean, and thus by extension

[7]The initial quota allocation is irrelevant for entry decisions as long as a friction-less quota trading market exists, which we assume. The initial allocation is of interest insofar as it determines the distribution of quota rent to the initial quota recipients.

does not know its efficiency rank.

Our incomplete information scenario begins with Nature picking θ from a uniform distribution on a subset of $\mathbf{R}_+$. Firms do not observe Nature's pick. Once picked by Nature, θ is fixed and a parameter of the model, although unknown to individual firms.

Together with the common knowledge assumption a firm's own observed θ_i provides a noisy signal about the true θ. Firms posterior beliefs are Bayesian. Thus, conditional on θ_i, firm i believes that θ is uniformly distributed over $[\theta_i - \varepsilon,\ \theta_i + \varepsilon]$. This implies, in particular, $E[\theta \mid \theta_i] = \theta_i$. We describe this scenario as one of *placement uncertainty*.

Hereafter, we denote the mass of firms that choose to enter the quota-regulated industry as A. The central question of the paper is, how does placement uncertainty affect a firm's entry decision and consequently the mass of entrants? Note that non-optimal industry performance manifests as excess or insufficient entry and/or entry of a cost inefficient subset of firms relative to full information.

3.1 Post entry quota market

Our base model is a single period two-stage game. In the first stage firms observe their θ_i-s and choose whether to enter (commit their capital) or stay out. Firms also draw up their quota buy/sell plans. In the second stage, the quotas are traded and equilibrium price determined. Firms execute their production plans and payoffs are realized.

The subgame perfect Nash equilibrium (or perfect Bayesian equilibrium under the incomplete information scenario) is thus also a rational expectations equilibrium. Firms anticipate the stage two quota price and capital rent they will receive when making stage one entry decisions.

Firms with positive quota endowments $w_i > 0$ that choose not to enter will sell their quota at any positive price. Their profit maximizing trading rule is simple: $v_i = -w_i$ for $r > 0$.

Entrants choose v_i to maximize operating profits (revenue minus cost of production) plus net permit trading receipts. Conditional on the quota price r, the entrants' stage two profit maximization problem is

$$\Pi = \max_{v_i} \Big\{ p(w_i + v_i) - \theta_i(w_i + v_i) - \frac{1}{2}\lambda(w_i + v_i)^2 - rv_i \Big\}, \tag{2}$$

subject to the constraint $w_i + v_i \geq 0$. The Lagrangian for this problem is,

$$L = p(w_i + v_i) - \theta_i(w_i + v_i) - \frac{1}{2}\lambda(w_i + v_i)^2 - rv_i - \mu(-w_i - v_i),$$

where μ is the Lagrangian multiplier. As v_i can be of any sign, the necessary conditions for an optimum are:

$$p - r - \theta_i - \lambda(w_i + v_i) + \mu = 0, \tag{3a}$$

$$w_i + v_i \geq 0, \ \ \mu(w_i + v_i) = 0, \tag{3b}$$

$$\mu \geq 0. \tag{3c}$$

From (3a) and (3b) we derive the net quota demand schedule for firm i that has entered the market:

$$v_i = \begin{cases} \frac{1}{\lambda}(p - r - \theta_i) - w_i & \text{for } 0 \le r \le p - \theta_i \\ -w_i & \text{for } p - \theta_i < r. \end{cases} \tag{4}$$

The above emphasizes the possibility that (post-entry) capital rent can be zero for a high cost entrant, i.e., if for firm i, the marginal profit from the first unit produced, $p - \theta_i$, is less than the quota price r, production and post entry variable profit will be zero. The schedule in (4) shows further that if $r > p - \theta_i$ entering firm i will minimize its loss by selling its quota and leaving its capital idle. The quota allocation satisfies the condition whereby marginal profit and marginal cost $\lambda q_i + \theta_i = \lambda(v_i + w_i) + \theta_i$ are equalized across all entrants with positive quota/production.

Quota demand schedules are combined to determine the market clearing quota price. Carrying out this derivation obtains,

$$r^* = r^*(A) = \max\left\{ p - \frac{\int_{i \in A} \theta_i d\theta_i}{A} - \frac{\lambda Q}{A}, 0 \right\}, \tag{5}$$

where A is endogenously determined. The notation $r^*(A)$ emphasizes that the equilibrium permit price depends on the set of entrants. The first term in the maximand in (5) is the mean marginal profit from producing a unit of quota. Expression $\frac{\int_{i \in A} \theta_i d\theta_i}{A} + \frac{\lambda Q}{A}$ is the mean marginal cost of entering firms where $\int_{i \in A} \theta_i d\theta_i / A$ is the mean cost efficiency among the set of entrants. We henceforth denote the mean cost efficiency among entering firms by $\bar{\theta}(A)$ (as opposed to θ which is the population mean).

It is clear that mean marginal costs increase, and thus r^* declines, with λ, Q, and the mean efficiency $\bar{\theta}(A)$. The effect of a change in A on r^* operates through two channels: mean efficiency and the share of total Q produced by each entering firm. This in turn implies that both A and the composition of cost efficiency of entering firms matter.

Combining (4) and (5) determines optimal production quantity for entering firm i:

$$q_i = w_i + v_i = \begin{cases} \frac{1}{\lambda}\left(\bar{\theta}(A) - \theta_i + \lambda Q / A\right) & \text{for } r^* > 0 \\ \frac{1}{\lambda}(p - \theta_i) & \text{for } r^* = 0 \end{cases}$$

The quantity produced by firm i is increasing in its relative cost efficiency, $\overline{\theta}(A) - \theta_i$, and in the total quota relative to the size of the set of entering firms, Q/A. In particular, q_i is independent of w_i.

Substituting (4) into (2), the entrant's total profit can be expressed as,

$$\Pi = \begin{cases} \underbrace{\frac{1}{2\lambda}(p - r^* - \theta_i)^2}_{\pi(r^*, \mathcal{I}_i)} + r^* w_i & \text{for } 0 \le r^* \le p - \theta_i \\ r^* w_i & \text{for } p - \theta_i < r^*. \end{cases} \tag{6}$$

Equation (6) shows that an entrant's total profit can be notionally decomposed into two parts. The first part denoted by $\pi(r^*, \mathcal{I}_i) = \frac{1}{2\lambda}(p - r^* - \theta_i)^2$ is the capital quasi-rent and is zero if the equilibrium permit price is too high and the entrant chooses to idle its capital. The second part is the quota endowment rent.

Note that capital rent $\pi(r^*, \mathcal{I}_i)$ is function of the parameters making up the firm's information set and equilibrium permit price which in turn depends on the endogenous set of entrants. To keep notations simple however, we henceforth suppress these dependencies.

Equation (6) further shows that π depends on the difference $p - r^*$ rather than on p itself. Substituting the equilibrium quota price from (5) (for the case where $r^* > 0$) obtains the following expression for the sum of the capital and quota rents.

$$\Pi = \frac{1}{2\lambda}\Big(\bar{\theta}(A) - \theta_i + \lambda Q/A\Big)^2 + \underbrace{\Big(p - \bar{\theta}(A) - \lambda Q/A\Big)}_{r^*(A)} w_i. \tag{7}$$

Equation (7) shows that the return to firm i's capital increases with its relative cost efficiency, $\bar{\theta}(A) - \theta_i$. Firm i's capital rent is highest when A is made up of a small number of higher cost rivals.

3.2 Enter or stay out

We can write the entry strategy for firm i as:

$$\left\{\begin{array}{c}\text{Enter}\\ \text{Stay out}\end{array}\right\} \text{ as } \quad E_i\left\{(\pi + r^* w_i)\right\} \quad \left\{\begin{array}{c}\geq\\ <\end{array}\right\} \quad \delta + E_i\left\{r^* w_i\right\}, \tag{8}$$

where $E_i\{.\} \equiv E\{.|\mathcal{I}_i\}$ denotes the expectations operator conditional on the firms' information set. The above expression shows that firm i expects to earn quota rent $r^* w_i$ regardless of whether it enters or stays out. The entry strategy therefore reduces to a firm entering if its expected capital rent $E\{\pi|\mathcal{I}_i\} \geq \delta$.[8]

The next sections present our main results. Entry and efficiency will vary across the full parameter space of the model, i.e., with information, cost structural parameters $(\theta, \varepsilon, \lambda, \delta)$, and with the demand and regulatory parameters, p and Q, respectively. Space limitations do not allow an exhaustive characterization of results for the entire parameter space. We focus instead on parameters recognized as significant determinants of industry structure and performance. We differentiate cases with low and high capital costs throughout.

4 Entry under full information

This section assumes that firms are fully informed about the mean cost efficiency of the population; $\mathcal{I}_i = (\theta_i, \theta, \varepsilon)$ for all i. A natural entry rule is a threshold or "switching" strategy where firm i enters if and only if its cost efficiency is less than or equal to a threshold, which we denote θ^*. We are particularly interested in a symmetric equilibrium in pure strategies in which all firms enter (with probability one) if $\theta_i \leq \theta^*$ and stay out otherwise.

[8] Montgomery (1972), in a model with cost heterogeneous firms but without capital, shows that when quota trade is frictionless, market efficiency is independent of an initial quota allocation. Although not the main focus of our paper this result, that with frictionless quota trade initial quota allocations do not impact physical capital investment, has not appeared in earlier literature.

Since θ_i is uniformly distributed over $[\theta - \varepsilon, \theta + \varepsilon]$, under an equilibrium threshold θ^* (if it exists) and for a given θ, the proportion of entering firms, which we denote $\alpha(\theta, \theta^*)$, is as follows,

$$\alpha(\theta, \theta^*) = \begin{cases} 0, & \text{if } \theta > \theta^* + \varepsilon, \\ \frac{1}{2\varepsilon} \int_{\theta-\varepsilon}^{\theta^*} d\theta_i = \frac{\theta^* - (\theta - \varepsilon)}{2\varepsilon}, & \text{if } \theta \in [\theta^* - \varepsilon, \theta^* + \varepsilon], \\ 1, & \text{if } \theta < \theta^* - \varepsilon. \end{cases} \tag{9}$$

If the threshold θ^* is below the population lower bound $\theta - \varepsilon$, no firms enter; $\alpha(\theta, \theta^*) = 0$ and $A = \emptyset$. If the threshold is particularly high, all firms enter.

The corresponding mean cost efficiency of entering firms for $\alpha(\theta, \theta^*) > 0$ is,

$$\bar{\theta} = \begin{cases} \frac{\frac{1}{2\varepsilon} \int_{\theta-\varepsilon}^{\theta^*} \theta_i d\theta_i}{\alpha(\theta, \theta^*)} = \frac{\theta^* + \theta - \varepsilon}{2}, & \text{if } \theta^* - \varepsilon \le \theta \le \theta^* + \varepsilon, \\ \theta, & \text{if } \theta < \theta^* - \varepsilon. \end{cases}$$

Then, (5), (9), and (4) can be used to express the equilibrium quota price as a function of the exogenous θ and the endogenous θ^* as follows:

$$r^* = \begin{cases} \max\{p - \frac{\theta^* + \theta - \varepsilon}{2} - \frac{2\varepsilon\lambda Q}{\theta^* - (\theta - \varepsilon)}, 0\}, & \text{for } \theta^* - \varepsilon \le \theta \le \theta^* + \varepsilon, \\ \max\{p - \theta - \lambda Q, 0\}, & \text{for } \theta < \theta^* - \varepsilon. \end{cases} \tag{10}$$

Proposition 2 shows the existence of a Nash equilibrium in pure strategies under full information.

Proposition 2. *For $\mathcal{I}_i = (\theta_i, \theta, \varepsilon)$, a threshold θ^{*F} exists such that firm i enters if $\theta_i \le \theta^{*F}$ and stays out, otherwise. Depending on the parametric configuration, θ^{*F} takes one of the two forms below.*

Case I. When $\delta \le \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, $\theta^{*F} = p - \sqrt{2\lambda\delta}$ and is independent of θ. The equilibrium quota price is given by

$$r^* = \begin{cases} p - \lambda Q - \theta \ge 0 & \textit{for } \theta \le p - \lambda Q \\ 0, & \textit{for } \theta > p - \lambda Q \end{cases}$$

Case II. When $\delta \ge \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$,

$$\theta^{*F} = \begin{cases} (\theta - \varepsilon) + \sqrt{2\lambda\delta + 4\varepsilon\lambda Q} - \sqrt{2\lambda\delta} & \textit{for } \theta \le \hat{\theta} \\ p - \sqrt{2\lambda\delta} & \textit{for } \theta \ge \hat{\theta} \end{cases}$$

where

$$\hat{\theta} = p + \varepsilon - \sqrt{2\lambda\delta + 4\varepsilon\lambda Q}.$$

Specifically, θ^{*F} is increasing for $\theta \le \hat{\theta}$ and constant for $\theta > \hat{\theta}$. The equilibrium quota price is given by

$$r^* = \begin{cases} p + \varepsilon - \sqrt{2\lambda\delta + 4\varepsilon\lambda Q} - \theta \ge 0 & \textit{for } \theta \le \hat{\theta} \\ 0, & \textit{for } \theta \ge \hat{\theta} \end{cases}$$

Proof: See appendix 8.2.

REMARK 1. Proposition 2 separates the parameter space into two regions: for case I, i.e., $\delta < \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, all firms can potentially enter; for case II, i.e., $\delta \geq \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, firms on the higher cost end of distribution never enter. For each case, the space of θ demarcates regions in which the quota constraint binds and the space in which it is slack.

REMARK 2. Entry decisions/actions are strategic substitutes when the quota binds. The payoff interaction operates through the equilibrium net price $p - r^*$, rather than an inelastic consumer demand for industry output. For case I, this requires $\theta \leq p - \lambda Q$ and for case II, this requires $\theta \leq \hat{\theta}$. Entry decisions are nonstrategic when the quota is in excess supply. This requires $\theta > p - \lambda Q$ and for case I, and $\theta > \hat{\theta}$ for case II. In such cases, capital rent for firm i depends on its absolute efficiency, θ_i, but not on its relative efficiency.

REMARK 3. Case I states that if capital costs are very low and fall below the cutoff, $\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, there is enough *room* for all firms to enter when $\theta < p - \lambda Q$. As θ rises above $p - \lambda Q$, and as profits decline with a rise in θ, some high cost firms choose to stay out and the quota is in excess supply.

As in the two firm case, the cutoff value of $\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$ is interpreted as the lowest rent that the highest cost firm earns when all firms enter and the quota binds. The highest cost firm in the population attains efficiency, $\theta + \varepsilon$. The quota price is $r^* = p - \lambda Q - \theta$ and therefore the quota binds for all $\theta < p - \lambda Q$. The variable profit of a firm i is $\frac{1}{2}\lambda q_i^2$ (as in the two firm example). The highest cost firm's $q_i = Q - \frac{\varepsilon}{\lambda}$. Thus, $\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, is a *minimum return* to vested capital that any firm can earn by entering.

REMARK 4. For case II, with capital costs above the cutoff $\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, the equilibrium threshold θ^{*F} and quota price r^* are piecewise linear functions of θ. In the region in which entry decisions are strategic, the entry threshold θ^{*F} is one-to-one increasing in θ, and thus the mass of entrants remains constant. In turn, the mean cost parameter, $\frac{\int_{i \in A} \theta_i d\theta_i)}{A}$, rises one-to-one and the permit price declines one-to-one with a rise in θ (see equation (5)). For the pivotal firm, the decline in the permit price is fully offset by the rise in θ^{*F}, thus ensuring that its capital rent remains at δ. All the lower cost firms continue to earn the same higher capital rents, whereas a fixed set of firms with $\theta_i \in \left(\theta^{*F}, \theta + \varepsilon\right)$ decide to stay out and sell their quota endowments.

When the quota constraint binds, i.e., $\theta < \hat{\theta}$, capital allocations are strategic substitutes. With a continuum of heterogenous cost firms, the pivotal firm earns capital rent equal to δ. Unlike the two firm example where only one firm enters, and the other stays out, irrespective of the value of δ, with a continuum of firms the mass of entrants is decreasing in δ.

Similar to the two firm model, entry decisions are strategically independent when the quota constraint is slack. Therefore, the entry threshold no longer depends on θ for $\theta \geq \hat{\theta}$ when $r^* = 0$. However, unlike in the two-firm case where the low cost firm drops out only when $\theta > p - \sqrt{2\lambda\delta} + \varepsilon$, with a continuum of heterogeneous firms a higher mass at the upper end of the cost distribution drops out as θ rises: the mass of entrants continuously shrinks from $\frac{\sqrt{2\lambda\delta + 4\varepsilon\lambda Q} - \sqrt{2\lambda\delta}}{2\varepsilon}$ to zero as θ rises from $\hat{\theta}$ to $p - \sqrt{2\lambda\delta} + \varepsilon$.

5 Entry under placement uncertainty

This section assumes that for all $i \in S$, $\mathcal{I}_i = \{\theta_i, \varepsilon\}$. Firm i forms a Bayesian posterior on θ. Specifically, conditional on θ_i, firm i believes that the unknown θ is uniformly distributed over $[\theta_i - \varepsilon, \theta_i + \varepsilon]$ and $E[\theta \mid \theta_i] = \theta_i$. We study the existence of a symmetric perfect Bayesian Nash equilibrium in which all firms adopt an identical threshold strategy with common threshold value denoted by θ^{*I}.

Firms must now consider the *expected* competition, quota price, and capital rent that will be realized post entry. We assume risk neutral firms rely on *expected* variable profits for making entry decisions. Let $\pi\left(\theta|\theta_i, \theta^{*I}\right)$ denote the variable profit of a firm with cost parameter θ_i, when the average cost is θ and the equilibrium entry threshold is θ^{*I}. Let $\pi\left(\theta_i, \theta^{*I}\right) \equiv E[\pi\left(\theta|\theta_i, \theta^{*I}\right)]$ define firm i's expected profit conditional on $\left\{\theta_i, \theta^{*I}\right\}$. Thus, firm i enters if

$$\pi\left(\theta_i, \theta^{*I}\right) = \frac{1}{2\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon} \pi\left(\theta|\theta_i, \theta^{*I}\right) d\theta = \frac{1}{2\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon} \frac{1}{2\lambda}\left(p - r(\theta, \theta^{*I}) - \theta_i\right)^2 d\theta \geq \delta,$$

and stays out otherwise.

Capital rent for firm i depends on the actual value of θ and takes one of the following forms depending on the mass of entrants $\alpha\left(\theta, \theta^{*I}\right)$ and permit price function, $r(\theta, \theta^{*I})$:

$$\begin{array}{lll} (a) & \alpha\left(\theta, \theta^{*I}\right) = 1,\ r(\theta, \theta^{*I}) > 0: & \pi(\theta|\theta_i, \theta^{*I}) = \frac{1}{2\lambda}\left(\theta + \lambda Q - \theta_i\right)^2 \\ (b) & \alpha\left(\theta, \theta^{*I}\right) < 1,\ r(\theta, \theta^{*I}) > 0: & \pi(\theta|\theta_i, \theta^{*I}) = \frac{1}{2\lambda}\left(\frac{\theta^{*I}+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{\theta^{*I}-(\theta-\varepsilon)} - \theta_i\right)^2 \\ (c) & \alpha\left(\theta, \theta^{*I}\right) < 1,\ r(\theta, \theta^{*I}) = 0: & \pi(\theta|\theta_i, \theta^{*I}) = \frac{1}{2\lambda}\left(p - \theta_i\right)^2 \end{array} \quad (11)$$

5.1 Equilibrium characterization

A firm with efficiency $\theta_i = \theta^{*I}$ is indifferent between entering and staying out. Thus θ^{*I} is the solution to,

$$\frac{1}{2\varepsilon}\int_{\theta^{*I}-\varepsilon}^{\theta^{*I}+\varepsilon} \frac{1}{2\lambda}\left(p - r(\theta, \theta^{*I}) - \theta^{*I}\right)^2 d\theta = \delta.$$

We describe the firm with $\theta_i = \theta^{*I}$ as the pivotal firm.

An upper bound on ε is required to ensure that every firm that enters the market utilizes its capital for production. This is specified as Assumption 1.[9]

Assumption 1: $\varepsilon < \lambda Q$.

5.2 Equilibrium with low capital cost

When $\delta \in \left[0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2\right]$, an equilibrium is shown to exist in which for sufficiently low θ realizations, the quota binds and all firms enter and earn capital rent exceeding δ. When θ is sufficiently high the quota is

[9]In the absence of this assumption, it is possible for $E[\pi(\theta_i)] \geq \delta$ and yet $p - r(\theta, \theta^*) < \theta_i$ for firm i, for some value of θ, in which case it is optimal for the firm to idle its vested capital. Assumption 1 helps simplify analysis.

correctly expected to be slack and the higher cost firms decide to stay out. The expected payoff of the pivotal firm and the equilibrium threshold is given as the solution to,

$$\pi\left(\theta^{*I},\theta^{*I}\right)=\frac{1}{2\varepsilon}\int_{\theta^{*I}-\varepsilon}^{\theta^{*I}+\varepsilon}\frac{1}{2\lambda}\left(p-\theta^{*I}\right)^2 d\theta=\delta, \tag{12}$$

Proposition 3. *For $\delta\in\left[0,\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2\right]$, a unique pure strategy perfect Bayesian Nash equilibrium exists under which a firm i enters if $\theta_i\leq\theta^{*I}=p-\sqrt{2\lambda\delta}$, and stays out otherwise.*

Proof: See appendix 8.3.

Proposition 2 case 1 and proposition 3 show that entry thresholds are identical under full and incomplete information when the cost of capital is below the cutoff $\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$. In this case placement uncertainty has no effect on entry. This result closely follows the two firm example. For low realizations of $\theta\leq p-\lambda Q$, when all firms enter and the quota market binds, the highest cost firm ($\theta_i=\theta+\varepsilon$) earns $\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$. When a firm gets $\theta_i\geq p-\lambda Q+\varepsilon$ it faces no market interaction as in the case of complete information and its entry threshold is identical to complete information.

5.3 Equilibrium with high capital cost

When $\delta>\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$, the pivotal firm expects that if θ is sufficiently high the quota price will be zero. That is, the pivotal firm expects the permit price to attain a value of zero for some $\theta=\hat{\theta}<\theta^{*I}$, where $\hat{\theta}$ equates the first term in the first expression of (10) to zero. Specifically, $\hat{\theta}(\theta^{*I})=(p+\varepsilon)-\sqrt{(p-\theta^{*I})^2+4\varepsilon\lambda Q}\in\left(\theta^{*I}-\varepsilon,\theta^{*I}\right)$. Thus $\hat{\theta}(\theta^{*I})$ lacks a closed form unlike its full information counterpart $\hat{\theta}$.

Combining all observations above, the payoff function of the pivotal firm and consequently the equilibrium threshold, θ^{*I}, is given by the condition,

$$\frac{1}{2\varepsilon}\int_{\theta^{*I}-\varepsilon}^{\hat{\theta}(\theta^{*I})}\frac{1}{2\lambda}\Big(\frac{\theta^{*I}+\theta-\varepsilon}{2}+\frac{2\varepsilon\lambda Q}{\theta^{*I}-(\theta-\varepsilon)}-\theta^{*I}\Big)^2 d\theta+\frac{1}{2\varepsilon}\int_{\hat{\theta}(\theta^{*I})}^{\theta^{*I}+\varepsilon}\frac{1}{2\lambda}(p-\theta^{*I})^2 d\theta=\delta. \tag{13}$$

Proposition 4. *For $\delta\in\left(\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2,\frac{(\lambda Q)^2}{2\lambda}+\frac{\varepsilon^2}{6\lambda}\right]$, a symmetric Perfect Bayesian Nash equilibrium in switching strategies exists in which the threshold θ^{*I} satisfies (13). Moreover, $\theta^{*I}<p-\sqrt{2\delta\lambda}$.*

Proof: See appendix 8.3.

REMARK 5. The equilibrium thresholds under full and incomplete information differ when the cost of capital is above the critical level, $\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$. Condition (13) reveals that in this case, θ^{*I} does not have a closed form but is independent of θ since firms must make entry decisions without knowing θ.

REMARK 6. Although θ^{*I} is independent of θ, the mass of entrants under both the full and incomplete information scenarios, depends not only on the respective thresholds but also on the realized (but unobserved under the second scenario) value of θ as the next section shows.

6 Industry structure and performance under placement uncertainty

In this section we isolate the effects of placement uncertainty by contrasting entry thresholds and the mass of entrants under our two information scenarios. We focus on the case $\delta > \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$ as the entry thresholds differ only under this situation.

Recall from proposition 2 that the full information entry threshold θ^{*F} is an increasing and piece-wise linear function of θ whereas the threshold under incomplete information θ^{*I} is a constant, independent of θ. For low realizations of θ, the threshold θ^{*F} is less than θ^{*I} and that this inequality is reversed at high realizations of θ. This relationship is captured in figure 6. θ^{*I} is represented as a horizontal line. θ^{*F} is linear and increasing up to $\theta = \hat{\theta}$ and is constant and equal to $p - \sqrt{2\lambda\delta}$ for $\theta > \hat{\theta}$. Note in particular that for $\theta = \theta^* - \varepsilon$,

$$\theta^{*F}(\theta^{*I} - \varepsilon) = \theta^{*I} + \sqrt{2\lambda\delta + 4\varepsilon\lambda Q} - \sqrt{2\lambda\delta} - 2\varepsilon < \theta^{*I},$$

where the last inequality follows due to $\delta > \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$.

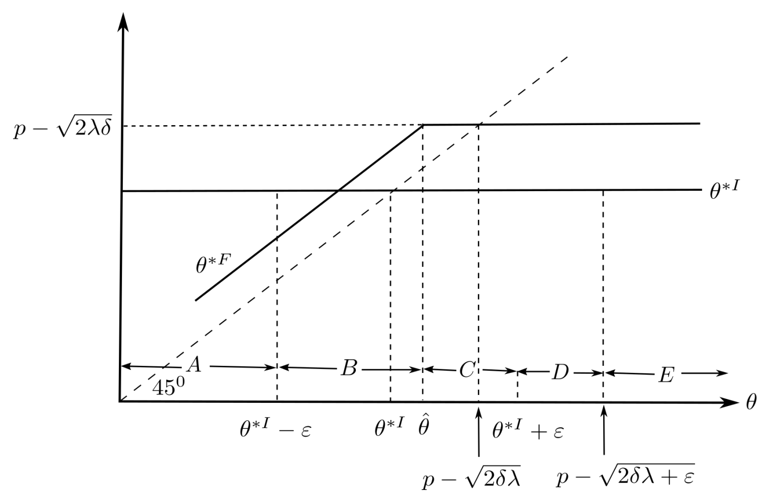


Figure 6: **Entry thresholds with and without placement uncertainty.**

Figure 6 partitions the space of θ into line segments A, B, C, D, and E. Equilibrium entry in each of these segments diverge in contrasting ways. There emerges another threshold value of $\theta = \tilde{\theta}$ in segment B, such that all θ realizations below this level cause excess entry, while all higher θ realizations lead to insufficient entry, under placement uncertainty. The next section maps figure 6 to the implied equilibrium entry under the two scenarios, and derives the extent of over- and under-entry under placement uncertainty *vis à vis* full information.

6.1 Mass of entrants under full and incomplete information

Following proposition 2 and equation (9), the mass of entrants under full information has the form,

$$\alpha^F = \begin{cases} \frac{\sqrt{2\delta\lambda+4\varepsilon\lambda Q}-\sqrt{2\delta\lambda}}{2\varepsilon} < 1 & \text{for } \theta \leq \hat{\theta} \\ \frac{(p-\sqrt{2\delta\lambda})-(\theta-\varepsilon)}{2\varepsilon} & \text{for } \theta \in \left(\hat{\theta}, p-\sqrt{2\delta\lambda}+\varepsilon\right) \\ 0 & \text{for } \theta \geq p-\sqrt{2\delta\lambda}+\varepsilon, \end{cases} \tag{14}$$

The mass of entrants under placement entry using equation (9) is given by,

$$\alpha^I = \begin{cases} 1 & \text{for } \theta \leq \theta^{*I}-\varepsilon \\ \frac{\theta^{*I}-(\theta-\varepsilon)}{2\varepsilon} & \text{for } \theta \in \left[\theta^{*I}-\varepsilon, \theta^{*I}+\varepsilon\right] \\ 0 & \text{for } \theta > \theta^{*I}+\varepsilon. \end{cases} \tag{15}$$

Figure 7 maps the difference $(\alpha^I - \alpha^F)$, which defines excess entry (EE for short) under placement uncertainty, as a piece-wise linear function over the domain indicated as segments A to E in figure 6. The details are offered in appendix 8.4.

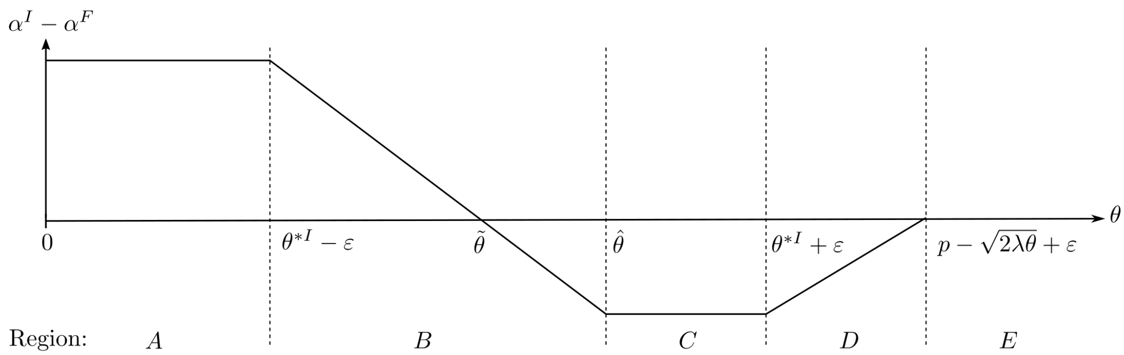
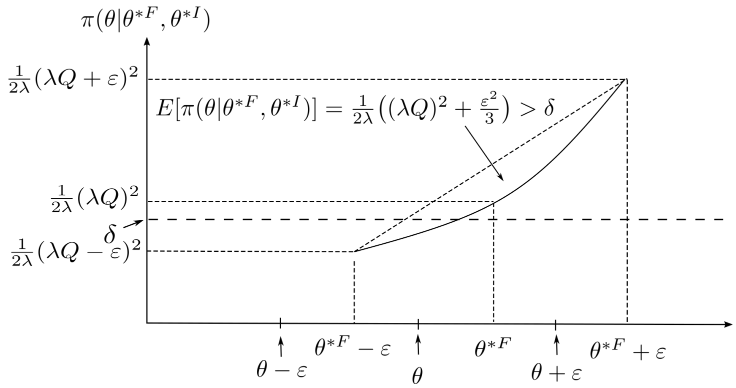


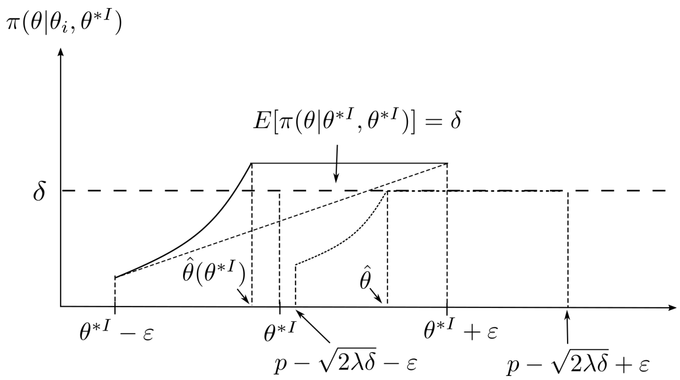
Figure 7: Entry Under Full and Incomplete Information, $\alpha^I - \alpha^F$

In figure 7, EE is positive at low levels of θ. In region A, where $\theta < \theta^{*I} - \varepsilon$, a low and increasing θ^{*F} translates into a constant $\alpha^F < 1$. Whereas, a relatively higher but constant level for θ^{*I} translates to $\alpha^I = 1$. This accounts for the constant and positive value for EE in region A.

For $\theta > \theta^{*I} - \varepsilon$, α^I is declining, whereas α^F remains constant for $\theta < \hat{\theta}$ (equation (14)). Hence EE falls and becomes negative at some point in region B. For $\theta \in \left(\hat{\theta}, \theta^{*I}+\varepsilon\right)$ in region C, both α^F and α^I decline in θ with an identical slope and therefore $\alpha^I - \alpha^F$ remains constant. In region D, $\theta \in \left(\theta^{*I}+\varepsilon, p-\sqrt{2\delta\lambda}+\varepsilon\right)$, $\alpha^I = 0$ whereas α^F remains positive but continues to decline. Finally at $\theta = p - \sqrt{2\delta\lambda} + \varepsilon$, $\alpha^F = 0$. Therefore, the entry difference rises from $-\alpha^F$ to 0 in region D and then remains 0 in region E.



(a) Low θ realization.



(b) High θ realization.

Figure 8: **Placement Uncertainty and capital rent.** Panels compare entry thresholds with and without placement uncertainty.

6.2 Capital payoff functions and entry thresholds

Proposition 5 formally summarizes the preceding excess/under entry results .

Proposition 5. *I. If $\delta \in \left[0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2\right]$, there is no excess entry of firms due to placement uncertainty.*

II. When $\delta > \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$, placement uncertainty: (a) causes excess entry over and above the full information benchmark, for $\theta \leq \tilde{\theta}$; (b) causes less entry than under full information for all $\theta > \tilde{\theta}$ for some $\tilde{\theta} = \theta^* + \varepsilon - \sqrt{2\lambda\delta + 4\epsilon\lambda Q} + \sqrt{2\delta\lambda} \in (\theta^* - \varepsilon, \theta^* + \varepsilon)$

For a given threshold and own cost efficiency, a firm's entry payoff is quasi-convex in θ; it is strictly increasing and convex in θ up to an endogenously determined value at which the permit price drops to zero (this consequently, varies across the two scenarios). The function is constant for values of θ higher than this level. Under full information, a firm is certain about post-entry quota competition and can assess the capital rent earned with entry, exactly. Under uncertainty, a firm forms Bayesian posterior on θ and ranks itself as the median cost firm. Based on this believe, the expected capital rent will vary with realizations of θ as follows.

First consider a low realization of θ such that the pivotal firm's cost parameter under complete information, θ^{*F}, is below $\theta^{*I} - 2\varepsilon$. Figure 8a displays the payoff function $\pi\left(\theta|\theta^{*F}, \theta^{*I}\right)$ of a firm with θ^{*F} as if this realization occurred under placement uncertainty. The firm considers itself as the median and believes $\theta \sim U\left[\theta^{*F} - \varepsilon, \theta^{*F} + \varepsilon\right]$. Since $\theta^{*F} < \theta^{*I} - 2\varepsilon$, the firm is certain that all firms are entering. After entering, if the firm is positioned at the higher end of the cost distribution, it will earn a rent below δ; if instead, it happens to be truly the median firm, it may earn a rent higher than δ, and of course it earns much above δ if it turns out to be on the lowest end of cost distribution. On the whole, it *expects* its capital rent will be higher than δ for two reasons. First, it considers itself as the median firm, although in truth θ^{*F} belongs to the upper tail of the cost distribution.[10] Second, the convexity of payoffs implies that $E\left[\pi\left(\theta|\theta^{*F}, \theta^{*I}\right)\right] > \pi\left(\theta^{*F}|\theta^{*F}, \theta^{*I}\right)$. In fact, its expected rent equals the upper bound of δ defined under proposition 4. Thus, an expected rent strictly higher than δ for a firm with θ^{*F} under incomplete information, implies a smaller but yet higher than δ capital rent for its neighboring higher cost firms with $\theta_i \in \left(\theta^{*F}, \theta + \varepsilon\right)$, and they also enter. These higher cost firms are precisely the firms that under full information choose to stay out.

Now consider the pivotal firm θ^{*I} under incomplete information. Figure 8b displays the payoff function $\pi\left(\theta|\theta^{*I}, \theta^{*I}\right)$. Since the pivotal firm expects the quota market to be slack for $\theta \in \left[\hat{\theta}\left(\theta^{*I}\right), \theta^{*I} + \varepsilon\right]$, its payoff is flat over this range. The capital payoff function over θ is therefore quasi-convex. Its expectation $E\left[\pi\left(\theta|\theta^{*I}, \theta^{*I}\right)\right] = \delta$ is what defines the firm with θ^{*I} as the pivotal firm. Any higher θ^{*I} will both depress and further stretch out the flat portion of payoff function, while keeping the payoff at $\theta^{*I} - \varepsilon$ at the same level. For example, suppose counter factually θ^{*I} is at $p - \sqrt{2\delta\lambda}$. Now the pivotal firm expects to earn only δ for those θ values that lie above $\hat{\theta}$, i.e, when the firm expects itself to be on the lower end of the cost distribution. On the other hand, for the states of θ towards the lower end of $\theta^{*I} - \varepsilon$, for which the pivotal firm is on the higher cost side, it expects rents lower than δ. The resultant payoff function is shown as a dashed line in figure 8b, which obviously rules out $\theta^{*I} = p - \sqrt{2\delta\lambda}$ as an equilibrium.

In contrast, under full information, for $\theta = \hat{\theta}$ the pivotal firm at $\theta^{*F} = p - \sqrt{2\delta\lambda}$ retains the same cost ranking and earns δ, same as what a pivotal firm earns for lower realizations of θ. All firms with cost parameter below θ^{*F} earn rents higher than δ. The result has the same flavor as in the two firm example under placement uncertainty. However, under full information, only one firm enters in the two firm example, and always earns rents higher than δ until it chooses to stay out. Under a continuum assumption a pivotal firm always earns δ.

6.3 Entry response to changes in Q and δ

The excess entry $\alpha^I - \alpha^F$ under placement uncertainty varies with model parameters. Of special interest are the regulatory cap Q and the capital cost parameter δ. We discuss how changes in these affect the equilibrium thresholds and mass of entrants. Under both scenarios, an increase in industry capacity, Q, provides room for more firms to enter whereas an increase in the capital cost, δ, restricts entry potentially. A detailed derivation of the results below is offered in appendix 8.5. Figures 9 (in appendix 8.5) shows how the entry thresholds

[10] Again, this result is similar to the two firm case, where for low values of θ the higher cost firm under full information does not enter, whereas under placement uncertainty each firm's payoff is an average of the low and high cost firms.

shift with a rise in Q or δ.

The incomplete information entry threshold θ^{*I} rises with a rise in Q. Under full information, when entry choices are strategic substitutes for low realizations of θ, an increase in Q increases α^F by raising θ^{*F}. But the entry threshold for higher values of θ, $p - \sqrt{2\lambda\delta}$ is independent of Q. As more firms enter under full information for lower θ values, excess entry is diminished but persists for a longer range of $\theta \leq \theta^{*I} - \varepsilon$. In contrast, the extent of inefficient under-entry is diminished over the range $\theta \in \left[\tilde{\theta}, p - \sqrt{2\lambda\delta} + \varepsilon\right]$.

A rise in capital cost δ decreases entry threshold θ^{*I}. It also lowers entry thresholds under full information *for all* θ realizations. Under placement uncertainty all firms enter for lower θ values but a smaller number of firms enter under full information. Excess entry is exacerbated, although over a lower range of $\theta \leq \theta^{*I} - \varepsilon$. Under-entry is diminished too because less firms enter under full information. The range over which under-entry exists, i.e., for $\theta \in \left[\theta^* + \varepsilon, p - \sqrt{2\lambda\delta} + \varepsilon\right]$ may rise or fall depending on whether the decline in θ^{*I} in response to a rise in δ is weaker or stronger than the fall in full information entry threshold.[11]

7 Conclusion

We study a two-stage entry game in a quota-regulated industry. Firms choose whether or not to commit capital to the industry under uncertainty over their productivity rank among a population of potential entrants. Firms form Bayesian beliefs about the distribution of productivity in the entrant population and thus, post entry competition for fixed production permits. Competition for quota (production permits) determines rents to the vested capital. We derive rational expectations, Bayesian Nash equilibrium entry rules and compare industry structure and performance with and without uncertainty over individual firm's relative productivity rank.

We find that placement uncertainty results in excess entry for parameterizations of our model that are likely to be observed in real world settings. Firms make entry decisions based on a belief that they achieve median productivity in the entrant population. They take investment risk based in an expected capital rent that is higher under placement uncertainty than under full information due to convexity of the capital payoff function. The result is excess entry and cost inefficient production relative to full information.

We show further that placement uncertainty can result in under-entry relative to full information. Here firms rationally overestimate the competition they could face post entry. This result arises when costs of production are high with expected capital rent bounded above. While inefficient *under entry* is possible, we expect the low cost scenario as more likely in real world settings, i.e., cap-and-trade regulations that address an overproduction externality are more likely in market settings with low costs and high profitability, the under-entry result is novel.

It can be shown that excess entry is exacerbated by placement *bias*.[12] When average industry costs are low, placement uncertainty and overplacement bias amplify the entry incentive, with observationally equivalent implications for industry structure and performance outcomes. Our results raise questions about the prevailing view on the role of overconfidence in market entry, capital investment, failed mergers, and over-trading of

[11] Formally, whether $\left|\frac{\partial\theta^*}{\partial\delta}\right| \lesseqgtr \sqrt{\frac{\lambda}{2\delta}}$.

[12] This result is shown formally in an extended appendix that is available from the authors upon request.

stocks. In particular, bias beliefs are commonly put forth as the main mechanism behind over-exuberance and ultimately inefficient entrepreneurial activity. Our finding that placement uncertainty is sufficient to generate excess entry warrants consideration, i.e., determining whether over-exuberant entry and investment is in fact driven by bias or uncertainty is a worthy topic of research.[13]

To this point we have not discussed policies to restore efficiency in market settings where placement uncertainty persists. Interventions to reduce competitive blind spots, e.g., exposing true productivity among firms is likely to be data intensive while requiring release of confidential firm-level information. Thus, an unlikely policy solution. Nudges are considered helpful in combatting decision bias and in exposing competitive blind spots. It should be noted further that we study a one shot entry game. The equilibrium quota permit price that obtains in our model, post entry, is informative about competitors true productivity. However, permit trading markets in CAT-regulated industries can be thin such that prices provide only noisy signals about competition. Learning can be slow allowing inefficiency to persist.[14] Policies that reduce trading frictions and improve the signal to noise ratio in permit trading prices may improve market performance.

[13]Experimental research may be able to isolate/separate the effects of uncertainty and bias on decision making in entrepreneurial settings.

[14]Asche et al. (2014) and Turner and Weninger (2005) report 10-20 year delays in the transition from overcapitalized to efficient fleet structures in fisheries that implement CAT regulations.

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8 Appendix: Proposition proofs

8.1 Proof of proposition 1

Nature first chooses $\theta \sim U\left[\underline{\theta}, \overline{\theta}\right]$; these bounds are assumed/set to be consistent with the equilibrium derived below. After θ is revealed by nature, the two firms are randomly assigned $\theta_i \in \{\theta_1, \theta_2\} = \{\theta - \varepsilon, \theta + \varepsilon\}$ with equal probability of $\frac{1}{2}$ each.

We will consider the following three information/timing scenarios:

1. Full information: Firms make entry choice after $\{\theta_1, \theta_2\}$ is revealed to each and it is commonly known.

2. Incomplete information with *placement* uncertainty: Firms realize their $\theta_i \in \{\theta_1, \theta_2\}$. However, firms know *only* their own θ_i. They do not know the other firm's $\theta_{j \neq i}$. The firms are Bayesian and therefore they assign equal probability to $\theta \in \{\theta_i - \varepsilon, \theta_i + \varepsilon\}$ in the range that is relevant for the following analysis.

8.1.1 Full information

Consider first the case when both firms enter and $r^* > 0$:

$$\theta_1 + \lambda q_1 = \theta_2 + \lambda\left(2Q - q_1\right);$$

The quotas utilized by the two firms are

$$q_1 = \frac{Q}{2} + \frac{\varepsilon}{\lambda}; q_2 = \frac{Q}{2} - \frac{\varepsilon}{\lambda}. \tag{16}$$

Their equilibrium variable profits are

$$\pi_1 = \frac{1}{2}\lambda q_1^2;\ \pi_2 = \frac{1}{2}\lambda q_2^2 \tag{17}$$

Since $p - r = \theta_1 + \frac{\lambda Q}{2} + \varepsilon = \frac{\lambda Q}{2} + \theta$, $r^* > 0$ iff $\theta < p - \frac{\lambda Q}{2}$.[15]

When the equilibrium $r^* = 0$, each firm equates

$$p = \theta_i + \lambda q_i, i = 1, 2.$$

In this case, the firm that enters gains a variable profit of

$$\pi_i = \frac{1}{2}\lambda q_i^2 = \frac{1}{2\lambda}\left(p - \theta_i\right)^2.$$

As firm 1's quota utilization is higher, so are its profits. It is the firm 2 that may choose not to enter while it may still be profitable for firm 1 to do so. Clearly, it all depends on θ.

[15] That $r < p$ only requires $\theta > -\lambda Q$, which holds by assumption on $\underline{\theta}$.

For $\theta < p - \frac{\lambda Q}{2}$, both firms will enter iff

$$\frac{1}{2}\lambda\left(\frac{Q}{2} - \frac{\varepsilon}{\lambda}\right)^2 > \delta \Longleftrightarrow \delta < \underline{\delta} \equiv \frac{1}{2\lambda}\left(\frac{\lambda Q}{2} - \varepsilon\right)^2$$

For $\delta < \underline{\delta}$ and $\theta > p - \frac{\lambda Q}{2}$, $r^* = 0$. To ensure that equilibria exist with $r^* > 0$, we assume that $\underline{\theta} < p - \frac{\lambda Q}{2}$. Both firms enter iff the firm of type 2 finds it profitable, i.e.,

$$\frac{1}{2\lambda}\left(p - \theta - \varepsilon\right)^2 \geq \delta \Longleftrightarrow \theta < p - \sqrt{2\lambda\delta} - \varepsilon$$

More efficient firm, type 1, will continue to enter as long as

$$\frac{1}{2\lambda}\left(p - \theta + \varepsilon\right)^2 \geq \delta \Longleftrightarrow \theta < p - \sqrt{2\lambda\delta} + \varepsilon$$

If $\delta > \underline{\delta}$ only firm 1 may enter if its profits exceed δ. Firm 1 continues to enter for θ realizations as long as

$$\frac{1}{2\lambda}\left(p - \theta_1\right)^2 > \delta \Longleftrightarrow \theta < p - \sqrt{2\lambda\delta} + \varepsilon.$$

To summarize, for $\delta \leq \underline{\delta}$ both firms enter if $\theta < p - \sqrt{2\lambda\delta} - \varepsilon$, only one firm will enter for $\theta \in \left(p - \sqrt{2\lambda\delta} - \varepsilon, p - \sqrt{2\lambda\delta} + \varepsilon\right)$. For $\delta > \underline{\delta}$ only firm 1 enters for $\theta_1 < \theta^{*F} \equiv p - \sqrt{2\lambda\delta}$.

Assumptions: Note that for $\delta = \underline{\delta}$, $\theta^{*F} + \varepsilon = p - \frac{\lambda Q}{2} + 2\varepsilon$. For all $\delta > \underline{\delta}, \theta^{*F} + \varepsilon < p - \frac{\lambda Q}{2} + 2\varepsilon$. Henceforth, we let $\bar{\theta} \in \left[p - \frac{\lambda Q}{2} + 2\varepsilon, p\right]$. We need $\frac{\lambda Q}{2} \geq 2\varepsilon$, to ensure that $\bar{\theta} \leq p$. Finally, to ensure $\theta_i > 0$, we let $\underline{\theta} > \varepsilon$. To summarize:

1. $\frac{\lambda Q}{2} \geq 2\varepsilon$.
2. $\underline{\theta} \in \left(\varepsilon, p - \frac{\lambda Q}{2}\right)$; $\bar{\theta} \in \left(p - \frac{\lambda Q}{2} + 2\varepsilon, p\right)$.
3. $\delta \in \left(\underline{\delta}, \bar{\delta}\right) = \left(\frac{1}{2\lambda}\left(\frac{\lambda Q}{2} - \varepsilon\right)^2, \frac{1}{2\lambda}\left(\left(\frac{\lambda Q}{2}\right)^2 + \varepsilon^2\right)\right)$.

8.1.2 Incomplete information with placement uncertainty

Under *placement uncertainty* a firm i knows its own realization of θ_i but it does not know whether its own type is 1 or 2. The firm forms Bayesian beliefs on various ex post equilibrium outcomes. With θ is unknown, all firms in equilibrium follow a cutoff strategy such that they enter iff $\theta_i < \theta^*$.

In equilibrium $\theta^{*I} \in \left(p - \frac{\lambda Q}{2} - \varepsilon, p - \frac{\lambda Q}{2} + \varepsilon\right)$ as we show below. This allows us to partition firm i's choice problem into the following domains of their own cost realizations:

I. $\theta_i \leq \theta^{*I} - 2\varepsilon \left(< p - \frac{\lambda Q}{2} - \varepsilon\right)$: Firm i knows that the other firm will also enter and the quota market will bind, because i is sure that $\theta < p - \frac{\lambda Q}{2}$. Its expected variable profit is

$$\pi_i^e = \frac{1}{2} * \frac{1}{2}\lambda q_1^2 + \frac{1}{2} * \frac{1}{2}\lambda q_2^2 = \bar{\delta} > \delta$$

II. $\theta_i \in \left(\theta^{*I} - 2\varepsilon, p - \frac{\lambda Q}{2} - \varepsilon\right)$. Again, i is sure that $\theta < p - \frac{\lambda Q}{2}$. It assigns $\frac{1}{2}$ probability to the other firm having entered, in which case i would be the higher cost firm with a profit $\frac{1}{2}\lambda q_2^2$. With $\frac{1}{2}$ probability it thinks the other firm has stayed out and i is the only entrant. Then its expected profit is

$$\begin{aligned} & \frac{1}{2} * \frac{1}{2}\lambda q_2^2 + \frac{1}{2} * \frac{1}{2}\lambda\left(p - \theta_i^2\right) \\ > \; & \frac{1}{4\lambda}\left[\left(\frac{\lambda Q}{2} - \varepsilon\right)^2 + \left(\frac{\lambda Q}{2} + \varepsilon\right)^2\right] = \bar{\delta}, \end{aligned}$$

where the inequality follows due to $\theta_i < p - \frac{\lambda Q}{2} - \varepsilon$. Thus, for all $\theta_i < p - \frac{\lambda Q}{2} - \varepsilon$ firm i enters.

III. If $\theta_i \in [p - \frac{\lambda Q}{2} - \varepsilon, p - \frac{\lambda Q}{2} + \varepsilon]$: Firm i assigns equal probabilities to $\theta \in \left\{\theta_i - \varepsilon, \theta_i + \varepsilon > p - \frac{\lambda Q}{2}\right\}$. If it thinks of itself as less efficient, i.e., of type 2, then $r > 0$. It rationally thinks that the other firm is entering as well and therefore assigns its own profits as $\frac{1}{2}\lambda q_2^2$. With an equal probability it considers itself to be more efficient and in that case $r = 0$ since $\theta_i + \varepsilon > p - \frac{\lambda Q}{2}$, and irrespective of whether the other firm enters or not, it assigns its own profits to be $\frac{1}{2\lambda}\left(p - \theta_i\right)^2$

$$\pi_i = \frac{1}{4\lambda}\left[\left(\frac{\lambda Q}{2} - \varepsilon\right)^2 + \left(p - \theta_i\right)^2\right]$$

Then, i will enter as long as $\theta_i < \theta^*$ defined by

$$\begin{aligned} \delta &= \frac{1}{4\lambda}\left[\left(\frac{\lambda Q}{2} - \varepsilon\right)^2 + \left(p - \theta^*\right)^2\right] \\ &\Rightarrow \theta^{*I} = p - \sqrt{4\lambda\delta - \left(\frac{\lambda Q}{2} - \varepsilon\right)^2} \end{aligned}$$

Thus no entry occurs for $\theta > \theta^{*I} + \varepsilon$.

It is easily checked that for $\delta = \underline{\delta}, \theta^{*I} = p - \frac{\lambda Q}{2} + \varepsilon$,[16] whereas for $\delta = \bar{\delta}, \theta^{*I} = p - \frac{\lambda Q}{2} - \varepsilon$.[17] Thus, $\theta^{*I} \in \left(p - \frac{\lambda Q}{2} - \varepsilon, p - \frac{\lambda Q}{2} + \varepsilon\right)$.

To summarize, both firms enter for $\theta \leq \theta^{*I} - 2\varepsilon$; only type 1 firm enters for $\theta \in \left(\theta^{*I}, \theta^{*I} + \varepsilon\right)$ .There is no entry for $\theta > \theta^{*I} + \varepsilon$.

8.1.3 $\theta^{*F} > \theta^{*I}$

The proof requires showing

$$\begin{aligned} \sqrt{2\lambda\delta - \varepsilon^2} &> \sqrt{2\lambda\delta} - \varepsilon \\ \text{or } \sqrt{2\lambda\delta - \varepsilon^2} + \varepsilon &> \sqrt{2\lambda\delta} \end{aligned}$$

[16] Then for all $\theta < \bar{\theta}$, both firms enter as under full information and common uncertainty.

[17] It can be show that for $\delta < \underline{\delta}$ the entry equilibrium will be the same as under full information.

Squaring up both sides and collecting terms to the LHS obtains

$$2\varepsilon\sqrt{2\lambda\delta - \varepsilon^2} - \varepsilon^2 + \varepsilon^2 > 0$$

8.2 Proof of Proposition 2: Equilibrium under complete information

The profit function for a firm with efficiency θ_i assumes two different forms, depending on whether equilibrium permit price is zero or non-zero:

$$\pi(\theta_i, \theta) = \begin{cases} \frac{1}{2\lambda}(p - \theta_i)^2, & \text{if for all } \theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon], r = 0 \\ \frac{1}{2\lambda}(p - r - \theta_i)^2, & \text{if for some } \theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon], r > 0 \end{cases}$$

Depending on the parametric configuration, there are two equilibria in a common threshold strategy.

8.2.1 Equilibrium with $\delta \in \left[0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2\right)$.

As Remark 3 explains, δ in this case is low enough to allow all firms to potentially enter at any positive r^*. As $r^* = p - \theta - \lambda Q$, $r^* \geq 0$ if $\theta \leq p - \lambda Q$. We claim that solution to $\frac{1}{2\lambda}(p - \theta^{*F})^2 = \delta$, namely, $\theta^{*F} = p - \sqrt{2\lambda\delta}$ is the equilibrium threshold strategy for this range of δ values.

The threshold $\theta^{*F} = p - \sqrt{2\lambda\delta}$ is equilibrium if the following conditions hold.

1. $\frac{1}{2\lambda}(p - r - \theta_i)^2 = \frac{1}{2\lambda}(\theta + \lambda Q - \theta_i)^2 > \delta$ for $\theta < p - \lambda Q$ and $\theta_i \in [\theta - \varepsilon, \theta + \varepsilon]$.
2. $\frac{1}{2\lambda}(p - \theta_i)^2 > \delta$ for $\theta \geq p - \lambda Q$, $\theta_i \in [\theta - \varepsilon, \theta + \varepsilon]$ and $\theta_i \leq p - \sqrt{2\lambda\delta}$.
3. $\frac{1}{2\lambda}(p - \theta_i)^2 < \delta$ for $\theta \geq p - \lambda Q$, $\theta_i \in [\theta - \varepsilon, \theta + \varepsilon]$ and $\theta_i > p - \sqrt{2\lambda\delta}$.

Condition (1) implies, $\frac{1}{2\lambda}(\theta + \lambda Q - \theta - \varepsilon)^2 = \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2 > \delta$. Since $\frac{1}{2\lambda}(p - \theta_i)^2$ is monotone decreasing in θ_i and has a zero at $\theta_i = p - \sqrt{2\lambda\delta}$, (2) and (3) are true.

8.2.2 Equilibrium with $\delta \geq \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$.

In this case, δ is high enough for some of the highest cost firms to always stay out. The equilibrium r^* takes the form in the first part of equation (10).

The candidate threshold θ^{*F} is determined by $\frac{1}{2\lambda}\left(\frac{\theta^{*F} + \theta - \varepsilon}{2} - \theta^{*F} + \frac{2\varepsilon\lambda Q}{\theta^{*F} - (\theta - \varepsilon)}\right)^2 = \delta$ which yields

$$\theta^{*F}(\theta) = (\theta - \varepsilon) \pm \sqrt{2\delta\lambda + 4\varepsilon\lambda Q} - \sqrt{2\delta\lambda}$$

Given that the equilibrium solution must satisfy, $\theta - \varepsilon \leq \theta^{*F}(\theta) \leq \theta + \varepsilon$, the first half of the inequality implies that the admissible solution is

$$\theta^{*F}(\theta) = (\theta - \varepsilon) + \sqrt{2\delta\lambda + 4\varepsilon\lambda Q} - \sqrt{2\delta\lambda}$$

The second half of the inequality, $\theta^{*F}(\theta) \leq \theta + \varepsilon$, gives us the admissible parametric configuration, $\delta \geq \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$ upon simplification.

Given the admissible form of $\theta^*(\theta)$, the admissible form for $r(\theta, \theta^*)$ is,

$$r(\theta,\theta^*) = \begin{cases} p + \varepsilon - \frac{\sqrt{2\lambda\delta+4\varepsilon\lambda Q}-\sqrt{2\lambda\delta}}{2} - \frac{2\varepsilon\lambda Q}{\sqrt{2\lambda\delta+4\varepsilon\lambda Q}-\sqrt{2\lambda\delta}} - \theta & \text{for } \theta < \hat{\theta} \\ 0, & \text{for } \theta > \hat{\theta} \end{cases}$$

where,

$$\begin{aligned} \hat{\theta} &= p + \varepsilon - \tfrac{\sqrt{2\lambda\delta+4\varepsilon\lambda Q}-\sqrt{2\lambda\delta}}{2} - \tfrac{2\varepsilon\lambda Q}{\sqrt{2\lambda\delta+4\varepsilon\lambda Q}-\sqrt{2\lambda\delta}} \\ &= p + \varepsilon - \sqrt{2\lambda\delta + 4\varepsilon\lambda Q} \end{aligned}$$

upon simplification.

Combining, the admissible threshold function is

$$\theta^*(\theta) = \begin{cases} (\theta - \varepsilon) + \sqrt{2\delta\lambda + 4\varepsilon\lambda Q} - \sqrt{2\delta\lambda} & \text{for } \theta < \hat{\theta} \\ p - \sqrt{2\lambda\delta} & \text{for } \theta \geq \hat{\theta} \end{cases}$$

For θ^{*F} to be equilibrium, the following conditions must be satisfied.

1. For $\theta < \hat{\theta}$

$$\begin{array}{ll} \frac{1}{2\lambda}\left(\frac{\theta^*(\theta)+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{\theta^*(\theta)-(\theta-\varepsilon)} - \theta_i\right)^2 > \delta & \text{for } \theta_i \leq \theta^*(\theta) \\ \frac{1}{2\lambda}\left(\frac{\theta^*(\theta)+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{\theta^*(\theta)-(\theta-\varepsilon)} - \theta_i\right)^2 < \delta & \text{for } \theta_i > \theta^*(\theta) \end{array}$$

2. For $\theta > \hat{\theta}$

$$\begin{array}{ll} \frac{1}{2\lambda}\left(p - \theta_i\right)^2 > \delta & \text{for } \theta_i \leq \theta^*(\theta) \\ \frac{1}{2\lambda}\left(p - \theta_i\right)^2 < \delta & \text{for } \theta_i > \theta^*(\theta) \end{array}$$

These conditions are satisfied because the profit functions are monotone decreasing in θ_i and has zeros at $\theta^*(\theta)$ for the relevant regions.

8.3 Proof of propositions 3 and 4: Equilibrium under incomplete information.

8.3.1 Proposition 3: Equilibrium with $\delta \leq \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$.

First note that there is no equilibrum in which the pivotal firm expects the premit price to be positive for some values of θ. For, suppose there is. At such values of θ the pivotal firm expects all firms to enter given that δ is no greater than the minimum capital rent earned by the highest cost firm. Hence such a θ^{*I} will be given by the solution of,

$$\frac{1}{4\lambda\epsilon}\Big[\int_{\theta^{*I}-\epsilon}^{p-\lambda Q}(\theta + \lambda Q - \theta^{*I})^2 d\theta + \int_{p-\lambda Q}^{\theta^{*I}+\epsilon}(p - \theta^{*I})^2 d\theta\Big]$$

In particular, this implies,

$$\frac{1}{2\lambda}\Big(\theta + \lambda Q - \theta^{*I}\Big)^2 \geq \frac{1}{2\lambda}\Big(\lambda Q - \epsilon\Big)^2 \geq \delta$$

for any δ in this range, and for values of $\theta \in [\theta^{*I} - \epsilon, p - \lambda Q]$. Furthermore,

$$\frac{1}{2\lambda}\Big(p - \theta^{*I}\Big)^2 \geq \frac{1}{2\lambda}\Big(\theta + \lambda Q - \theta^{*I}\Big)^2$$

for values of $\theta \in [p - \lambda Q, \theta^{*I} + \epsilon]$. Hence the equation does not have a solution.

This implies that the only equilibrium for this case is one which solves,

$$\pi(\theta^{*I}, \theta^{*I}) = \frac{1}{2\epsilon}\int_{\theta^{*I}-\epsilon}^{\theta^{*I}+\epsilon} \frac{1}{2\lambda}(p - \theta^{*I})^2 = \delta$$

namely, $\theta^{*I} = p - \sqrt{2\lambda\delta}$.

To show $\theta^{*I} = p - \sqrt{2\lambda\delta}$ is equilibrium we need to show that the expected profit, $\pi(\theta_i, \theta^{*I})$, is greater than δ when $\theta_i < \theta^{*I}$ and less than δ when $\theta_i > \theta^{*I}$. The expected profit function takes different forms depending on the zones in which θ_i lie.

1. For $\theta_i < \theta^{*I} - 2\varepsilon = p - \sqrt{2\lambda\delta} - 2\varepsilon$, a firm with efficiency θ_i expects all firms to enter and the quota to bind. Hence individual profit function must satisfy the condition,

   $$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta > \delta$$

   Note that

   $$\begin{aligned}
   \pi(\theta_i, \theta^{*I}) &= \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta \\
   &= \frac{1}{12\lambda\varepsilon}\Big[(\lambda Q + \varepsilon)^3 - (\lambda Q - \varepsilon)^3\Big] \\
   &= \frac{1}{12\lambda\varepsilon}(2\varepsilon)\Big((\lambda Q)^2 - \varepsilon^2 + 2(\lambda Q)^2 + 2\varepsilon^2\Big) \\
   &= \frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda} > \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2 \geq \delta
   \end{aligned}$$

   by Assumption 1. Hence the condition is satisfied.

2. Next consider values of θ_i such that $p - \sqrt{2\lambda\delta} - 2\epsilon = \theta^{*I} - 2\epsilon \leq \theta_i \leq p - \lambda Q + \epsilon$. The firm with cost parameter θ_i expects all firms to enter and the quota to bind for $\theta \leq p - \lambda Q$. Note further, $p - \lambda Q \leq \theta^{*I} - \epsilon$ implies $p - \lambda Q + \epsilon \leq \theta^{*I}$ which in turn implies $\theta_i \leq \theta^{*I}$. Hence the firm's profit function must satisfy,

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\epsilon}\int_{\theta_i-\epsilon}^{p-\lambda Q}(\theta+\lambda Q-\theta_i)^2 d\theta + \frac{1}{4\lambda\epsilon}\int_{p-\lambda Q}^{\theta_i+\epsilon}(p-\theta_i)^2 d\theta \geq \delta$$

The individual profit function simplifies to,

$$\begin{aligned}\pi(\theta_i, \theta^{*I}) &= \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{p-\lambda Q}(\theta-\theta_i+\lambda Q)^2 \, d\theta + \frac{1}{4\lambda\varepsilon}\int_{p-\lambda Q}^{\theta_i+\varepsilon}(p-\theta_i)^2 \, d\theta \\ &= \frac{1}{4\lambda\varepsilon}\left[\frac{(p-\theta_i)^3}{3} - \frac{(\lambda Q-\varepsilon)^3}{3} + (p-\theta_i)^2(\theta_i+\varepsilon-p+\lambda Q)\right]\end{aligned}$$

The right hand side equals $\frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda}$ when $\theta_i = p - \lambda Q - \varepsilon$, and $\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2 \geq \delta$ when $\theta_i = p - \lambda Q + \varepsilon$. Thus the value of the function is higher than δ at one of the end points and as the following derivative shows, the function is strictly declining.

The derivative of $\pi(\theta_i, I_{\theta^{*I}})$ with respect to θ_i equals

$$\begin{aligned}&\frac{1}{4\lambda\varepsilon}\left[-(\lambda Q-\varepsilon)^2 + (p-\theta_i)^2 - 2\int_{\theta_i-\varepsilon}^{p-\lambda Q}(\theta-\theta_i+\lambda Q)\, d\theta - 2\int_{p-\lambda Q}^{\theta_i+\varepsilon}(p-\theta_i)\, d\theta\right] \\ &= \frac{1}{4\lambda\varepsilon}\left[-(\lambda Q-\varepsilon)^2 + (p-\theta_i)^2 - (p-\theta_i)^2 + (\lambda Q-\varepsilon)^2 - 2\int_{p-\lambda Q}^{\theta_i+\varepsilon}(p-\theta_i)\, d\theta\right] \\ &= \frac{1}{4\lambda\varepsilon}\left[-2\int_{p-\lambda Q}^{\theta_i+\varepsilon}(p-\theta_i)\, d\theta\right] < 0\end{aligned}$$

The condition is therefore satisfied.

3. When $p - \lambda Q + \varepsilon < \theta_i < \theta^{*I}$, the individual profit function must satisfy,

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{2\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon}\frac{1}{2\lambda}(p-\theta_i)^2 \, d\theta = \frac{1}{2\lambda}(p-\theta_i)^2 \geq \delta.$$

The right hand side is strictly declining and greater than δ for $\theta_i < \theta^{*I}$, because at $\theta_i = \theta^{*I}$, $\frac{1}{2\lambda}(p-\theta_i)^2 = \delta$. Hence the condition is satisfied.

4. When $\theta_i > \theta^{*I}$, the individual profit function must satisfy,

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{2\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon}\frac{1}{2\lambda}(p-\theta_i)^2 \, d\theta < \delta.$$

This is true because the left hand side is strictly declining and has a value of δ at $\theta_i = \theta^{*I}$.

Thus $\theta^{*I} = p - \sqrt{2\lambda\delta}$ is equilibrium.

8.3.2 Proposition 4: Equilibrium with $\delta \in \left(\frac{1}{2\lambda}\left(\lambda Q - \varepsilon^2, \right), \frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda}, \right)$

For any given θ^{*I}, the quota price function, $r(\theta, \theta^{*I}) = p - \frac{\theta^{*I}+\theta-\varepsilon}{2} - \frac{2\varepsilon\lambda Q}{\theta^{*I}-(\theta-\varepsilon)}$ attains zero at a value of $\theta < \theta^{*I} + \varepsilon$, since p is finite and last term tends to infinity as $\theta \longrightarrow \theta^{*I} + \varepsilon$. Given θ^{*I}, the roots of $p - \frac{\theta^{*I}+\theta-\varepsilon}{2} - \frac{2\varepsilon\lambda Q}{\theta^{*I}-(\theta-\varepsilon)} = 0$ are given by $\hat{\theta}(\theta^{*I}) = (p+\varepsilon) \pm \sqrt{(p-\theta^{*I})^2 + 4\varepsilon\lambda Q}$. The restriction $\hat{\theta}(\theta^{*I}) < \theta^{*I} + \varepsilon$ implies that only the root $\hat{\theta}(\theta^{*I}) = (p+\varepsilon) - \sqrt{(p-\theta^{*I})^2 + 4\varepsilon\lambda Q}$ need be considered. It is also clear from previous discussions that if an equilibrium exists for $\delta > \frac{1}{2\lambda}\left(\lambda Q - \varepsilon\right)^2$, then the permit price is strictly positive at $\theta = \theta^{*I} - \varepsilon$, implying, $\hat{\theta}(\theta^{*I}) > \theta^{*I} - \varepsilon$.

It is straightforward to check that under Assumption 1, $\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2 < \frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda}$ and the set $\delta \in \left(\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2, \frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda}\right]$ is non-empty. We now proceed to the main steps of the proof of Proposition 4.

STEP I. We show that there is a unique solution θ^{*I} that satisfies condition (13). Consider the function,

$$\pi(k) = \frac{1}{2\varepsilon}\int_{k-\varepsilon}^{\hat{\theta}(k)} \frac{1}{2\lambda}\left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)} - k\right)^2 d\theta + \frac{1}{2\varepsilon}\int_{\hat{\theta}(k)}^{k+\varepsilon} \frac{1}{2\lambda}(p-k)^2 d\theta,$$

where $\hat{\theta}(k) = p + \varepsilon - \sqrt{(p-k)^2 + 4\varepsilon\lambda Q}$.

A change of variable $z = \theta - (k+\varepsilon)$ allows us to evaluate the first integral. With this change of variable, the upper and lower limits of the integration are, respectively, $\hat{\theta}(k) - k - \varepsilon = (p-k) - \sqrt{(p-k)^2 + 4\varepsilon\lambda Q}$ and -2ε. Evaluating the integral using the new variable and then substituting the new variable back and simplifying, we have,

$$\pi(k) = \frac{1}{4\lambda\varepsilon}\left[\begin{array}{c} -\frac{4(\varepsilon\lambda Q)^2}{(p-k)-\sqrt{(p-k)^2+4\varepsilon\lambda Q}} - \frac{4(\varepsilon\lambda Q)^2}{2\varepsilon} + \frac{\left((p-k)-\sqrt{(p-k)^2+4\varepsilon\lambda Q}\right)^3}{12} \\ +\frac{8\varepsilon^3}{12} - 2\varepsilon\lambda Q\left[(p-k) - \sqrt{(p-k)^2 + 4\varepsilon\lambda Q} + 2\varepsilon\right] \\ -\left[(p-k) - \sqrt{(p-k)^2 + 4\varepsilon\lambda Q}\right](p-k)^2 \end{array}\right] \tag{18}$$

We try to show next that $\frac{d\pi(k)}{dk} < 0$. A second change of variable helps us to do that. Define

$$x \equiv \sqrt{(p-k)^2 + 4\varepsilon\lambda Q} - (p-k) > 0,$$

and note that

$$\frac{dx}{dk} = 1 - \frac{p-k}{\sqrt{(p-k)^2 + 4\varepsilon\lambda Q}} > 0.$$

Further note that $(p-k)^2 = \frac{x^2}{4} + \frac{(2\varepsilon\lambda Q)^2}{x^2} - 2\varepsilon\lambda Q$. With the second change of variable, $\pi(k)$ can be rewritten as

$$\pi(k, I_k) = \frac{1}{4\lambda\varepsilon}\left[\begin{array}{c} -\frac{4(\varepsilon\lambda Q)^2}{2\varepsilon} + \frac{8\varepsilon^3}{12} - 2\varepsilon\lambda Q\,(2\varepsilon) \\ +\frac{x^3}{6} + \frac{8(\varepsilon\lambda Q)^2}{x} \end{array}\right]$$

Thus, whether $\pi(k)$ is increasing or decreasing in k depends on whether it decreases or increases in x.

$$\begin{aligned} \frac{d\pi}{dx} &= \frac{1}{4\lambda\varepsilon}\left[\frac{x^2}{2} - \frac{8(\varepsilon\lambda Q)^2}{x^2}\right] \\ &= \frac{1}{4\lambda\varepsilon}\left(\frac{x}{\sqrt{2}} + \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right)\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right) \end{aligned}$$

Thus the sign of $\frac{d\pi(k)}{dk}$ depends on the sign of $\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right)$. Substituting the expression for x back and simplifying, it is straightforward to check that so long as $(p-k) > 0$ (true for values of k we are interested in), $\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right) < 0$.

Thus $\frac{d\pi(k)}{dk} < 0$.

We next show that the function $\pi(k)$ is greater than δ for some k and less than δ for some k.

We begin by asserting that for any given k and $\theta \in [k-\varepsilon, k+\varepsilon]$, the following inequality is true.

$$(\theta + \lambda Q - k) \leq \left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)} - k\right) \leq (p-k) \tag{19}$$

The first inequality holds for the following reason. For any given k and $\theta \in [k-\varepsilon, k+\varepsilon]$, the expression,

$$\left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}\right) = \theta + \lambda Q \text{ when } \theta = k - \varepsilon.$$

Both functions are increasing in θ. The slope of $\theta + \lambda Q$ is 1. The slope of $\left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}\right)$ is $\left(\frac{1}{2} + \frac{2\varepsilon\lambda Q}{(k-(\theta-\varepsilon))^2}\right)$. As $\frac{2\varepsilon\lambda Q}{(k-(\theta-\varepsilon))^2} > \frac{4\varepsilon^2}{(k-(\theta-\varepsilon))^2}$ by Assumption 1 and $k-(\theta-\varepsilon) \leq 2\varepsilon$, the ratio $\frac{2\varepsilon\lambda Q}{(k-(\theta-\varepsilon))^2} > 1$. Hence $(\theta+\lambda Q - k) \leq \left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)} - k\right)$ for $\theta \in [k-\varepsilon, k+\varepsilon]$, for any given k.

The second equality follows from the fact that the expression, $\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}$, represents quasi rent on capital and is less than or equal to p.

We therefore have,

$$\frac{1}{2\varepsilon}\int_{k-\varepsilon}^{k+\varepsilon} \frac{1}{2\lambda}(\theta+\lambda Q-k)^2\,d\theta \leq \pi(k) \leq \frac{1}{2\varepsilon}\int_{k-\varepsilon}^{k+\varepsilon}\frac{1}{2\lambda}(p-k)^2\,d\theta. \tag{20}$$

Note that for $k = p - \lambda Q + \varepsilon$, $\frac{1}{2\lambda}(p-k)^2 = \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2$. Hence, $\pi(k) \leq \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2 < \delta$ for some value of k.

Similarly note that for $k = p - \lambda Q - \varepsilon$,

$$\frac{1}{2\varepsilon}\int_{k-\varepsilon}^{k+\varepsilon}\frac{1}{2\lambda}\left(\theta+\lambda Q-k\right)^2 d\theta = \frac{1}{12\lambda\varepsilon}\left[(\lambda Q+\varepsilon)^3 - \lambda Q-\varepsilon)^3\right] = \frac{\lambda Q^2}{2}+\frac{\varepsilon^2}{6\lambda} > \delta$$

Thus the function $\pi(k)$ is greater than δ for some k. Combining all observations, it has a unique intersection $k = \theta^{*I}$ with δ. Furthermore as $\pi(k)$ is decreasing in k, the intersection θ^{*I} is decreasing in δ.

STEP II. We next show that the switching strategy with the threshold θ^{*I} is an equilibrium. We need to show that for a firm with efficiency θ_i, $\pi(\theta_i, \theta^{*I}) > \delta$ for $\theta_i < \theta^{*I}$ and $\pi(\theta_i, \theta^{*I}) < \delta$ for $\theta_i > \theta^{*I}$.

The fact that θ^{*I} has no closed form makes the analysis relatively more complicated compared to what it is under proposition 3. We begin by characterizing the function $\pi(\theta_i, \theta^{*I})$ for different zones in which θ_i may lie, given the solution θ^{*I} and the condition it must satisfy.

1. For $\theta_i < \theta^{*I} - 2\varepsilon$ or $\theta_i + 2\varepsilon < \theta^{*I}$.

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon}\left(\theta+\lambda Q-\theta_i\right)^2 d\theta > \delta.$$

From the perspective of the θ_i-type, all possible types for any $\theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon]$ are below the threshold θ^{*I}. Thus, all firms enter for any $\theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon]$ and the individual profit function takes the above form. The expected profit of a firm with efficiency θ_i must be higher than δ.

2. For $\theta^{*I} - 2\varepsilon < \theta_i < \hat{\theta}(\theta^{*I}) - \varepsilon < \theta^{*I}$,

$$\begin{aligned}\pi(\theta_i, \theta^{*I}) &= \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta^{*I}-\varepsilon}\left(\theta-\theta_i+\lambda Q\right)^2 d\theta \\ &\quad+\frac{1}{4\lambda\varepsilon}\int_{\theta^{*I}-\varepsilon}^{\theta_i+\varepsilon}\left(\frac{\theta^{*I}+\theta-\varepsilon}{2}-\theta_i+\frac{2\varepsilon\lambda Q}{\theta^{*I}-(\theta-\varepsilon)}\right)^2 d\theta > \delta.\end{aligned}$$

A firm with efficiency θ_i does not expect all firms to be active for all $\theta \in [\theta_i - \varepsilon, \theta_i + \varepsilon]$. For the range, $\theta \in [\theta_i - \varepsilon, \theta^{*I} - \varepsilon]$, the firm expects all firms to enter, because the highest cost firm in this range has a cost parameter less than the threshold. For values of θ above $\theta^{*I} - \varepsilon$, some of the high cost firms are expected to have parameter values above the threshold and will not enter. The expected profit function therefore has two parts and must have a value higher than δ.

3. For $\hat{\theta}(\theta^{*I}) - \varepsilon < \theta_i < \theta^{*I}$, the profit function has three parts.

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i - \varepsilon}^{\theta^{*I} - \varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta$$
$$+ \frac{1}{4\lambda\varepsilon} \int_{\theta^{*I} - \varepsilon}^{\hat{\theta}(\theta^{*I})} \left(\frac{\theta^{*I} + \theta - \varepsilon}{2} - \theta_i + \frac{2\varepsilon\lambda Q}{\theta^{*I} - (\theta - \varepsilon)} \right)^2 d\theta$$
$$+ \frac{1}{4\lambda\varepsilon} \int_{\hat{\theta}(\theta^{*I})}^{\theta_i + \varepsilon} (p - \theta_i)^2 \, d\theta > \delta.$$

Similar arguments as in (2) explain the first two components of the function. The third component is explained by the fact that if $\theta > \hat{\theta}(\theta^{*I})$, the expected permit price is zero.

4. For, $\theta^{*I} < \theta_i < \hat{\theta}(\theta^{*I}) + \varepsilon < \theta^{*I} + 2\varepsilon$, the profit function has two parts.

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i - \varepsilon}^{\hat{\theta}(\theta^{*I})} \left(\frac{\theta^{*I} + \theta - \varepsilon}{2} - \theta_i + \frac{2\lambda\varepsilon Q}{\theta^{*I} - (\theta - \varepsilon)} \right)^2 d\theta$$
$$+ \frac{1}{4\lambda\varepsilon} \int_{\hat{\theta}(\theta^{*I})}^{\theta_i + \varepsilon} (p - \theta_i)^2 \, d\theta < \delta$$

The quota price is positive for some values of θ and zero for others, accounting for the two components. As $\theta_i > \theta^{*I}$, production must be less profitable for the θ_i-type, compared to the outside option.

5. For $\hat{\theta}(\theta^{*I}) + \varepsilon < \theta_i < \theta^{*I} + 2\varepsilon$,

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} (p - \theta_i)^2 \, d\theta < \delta.$$

It is straightforward to check that $\pi(\theta_i, \theta^{*I})$ is continuous in θ_i.
The rest of the proof, shows that the required inequality is satisfied for each zone.

1. When $\theta_i < \theta^{*I} - 2\varepsilon$,

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i - \varepsilon}^{\theta_i + \varepsilon} (\theta + \lambda Q - \theta_i)^2 \, d\theta = \frac{1}{12\lambda\varepsilon} \left[(\lambda Q + \varepsilon)^3 - \lambda Q - \varepsilon)^3 \right] = \frac{\lambda Q^2}{2} + \frac{\varepsilon^2}{6\lambda} > \delta$$

by Assumption 1 and the first inequality is satisfied.

2. When $\theta^{*I} - 2\varepsilon < \theta_i < \hat{\theta}(\theta^{*I}) - \varepsilon < \theta^{*I}$, the condition (20) shows that for any θ_i and given θ^{*I},

$$\pi(\theta_i, \theta^{*I}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i - \varepsilon}^{\theta^{*I} - \varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta + \frac{1}{4\lambda\varepsilon} \int_{\theta^{*I} - \varepsilon}^{\theta_i + \varepsilon} \left(\frac{\theta^{*I} + \theta - \varepsilon}{2} - \theta_i + \lambda \frac{2\varepsilon Q}{\theta^{*I} - (\theta - \varepsilon)} \right)^2 d\theta$$

$$\geq \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon}(\theta+\lambda Q-\theta_i)^2\,d\theta > \delta.$$

Thus the second inequality is also true.

3. Note that $\pi(\theta_i,\theta^{*I})$ is not only continuous but differentiable (in fact, twice differentiable) over the interval $\hat{\theta}(\theta^{*I})-\varepsilon<\theta_i<\theta^{*I}$ as well. From item 2 above, at $\theta_i=\hat{\theta}(\theta^{*I})-\varepsilon$, $\pi(\theta_i,\theta^{*I})>\delta$.

   By continuity, as $\theta_i \longrightarrow \theta^{*I}$, the function $\pi(\theta_i,\theta^{*I})$ converges to the function $\pi(\theta^{*I})$ pointwise. Hence, $\pi(\theta_i,\theta^{*I})\longrightarrow\delta$ as $\theta_i\longrightarrow\theta^{*I}$.

   Moreover, STEP I shows that $\pi(\theta^{*I})$ is declining at $\theta_i=\theta^{*I}$. Hence the slopes of the two functions must also converge as $\theta_i\longrightarrow\theta^{*I}$ and in particular $\pi(\theta_i,\theta^{*I})$ must be declining at $\theta_i=\theta^{*I}$. Thus, $\pi(\theta_i,\theta^{*I})$ must have at least one stationary point that is a maximum in this interval.

   We therefore check the roots of the derivative of $\pi(\theta_i,\theta^{*I})$ with respect to θ_i.

   The derivative of the first term of $\pi(\theta_i,\theta^{*I})$ with respect to θ_i is given by (using Leibnitz's rule),

$$\begin{aligned}
&\frac{-1}{2\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta^{*I}-\varepsilon}(\theta-\theta_i+\lambda Q)\,d\theta-(\lambda Q-\varepsilon)^2\\
&=-\left(\theta^{*I}-\theta_i+\lambda Q-\varepsilon\right)^2+(\lambda Q-\varepsilon)^2-(\lambda Q-\varepsilon)^2\\
&=-\left(\theta^{*I}-\theta_i+\lambda Q-\varepsilon\right)^2.
\end{aligned}$$

   The derivative of the second term is given by,

$$\begin{aligned}
&-\frac{1}{2\lambda\varepsilon}\int_{\theta^{*I}-\varepsilon}^{\hat{\theta}(\theta^{*I})}\left(\frac{\theta^{*I}+\theta-\varepsilon}{2}-\theta_i+\lambda\frac{2\varepsilon Q}{\theta^{*I}-(\theta-\varepsilon)}\right)d\theta\\
&=2\left(\theta^{*I}-\varepsilon\right)^2-\frac{\left(\theta^{*I}+\hat{\theta}-\varepsilon\right)^2}{2}+4\varepsilon\lambda Q\left(\ln\left[\theta^{*I}+\varepsilon-\hat{\theta}\right]-\ln\left[2\varepsilon\right]\right)+2\theta_i\left(\hat{\theta}\left(\theta^{*I}\right)-\left(\theta^{*I}-\varepsilon\right)\right).
\end{aligned}$$

   The derivative of the third term is given by

$$\frac{1}{4\lambda\varepsilon}\left[3\left(p-\theta_i\right)^2-2\left(p-\theta_i\right)\sqrt{\left(p-\theta^{*I}\right)^2+4\varepsilon\lambda Q}\right]$$

   These terms of the derivatives can be combined to get

$$\frac{1}{2\lambda\varepsilon}\left[\theta_i^2-2\left(p-\frac{\lambda Q+\varepsilon}{2}\right)\theta_i+\Omega\left[\theta^{*I}\right]\right] \tag{21}$$

where

$$\Omega\left[\theta^{*I}\right] \equiv \left(\theta^{*I}-\varepsilon\right)^2 - \frac{\left(\hat{\theta}\left(\theta^{*I}\right)+\theta^{*I}-\varepsilon\right)^2}{4} + 2\varepsilon\lambda Q \ln\left[1-\frac{\hat{\theta}\left(\theta^{*I}\right)-\left(\theta^{*I}-\varepsilon\right)}{2\varepsilon}\right]$$
$$+\frac{p^2-\left(\lambda Q+\theta^{*I}-\varepsilon\right)^2}{2} + p\left(\hat{\theta}\left(\theta^{*I}\right)-\varepsilon\right).$$

Derivative (21) has two roots given by

$$p-\frac{\lambda Q+\varepsilon}{2} \pm \sqrt{\left(p-\frac{\lambda Q+\varepsilon}{2}\right)^2-\Omega\left[\theta^{*I}\right]}$$

Note that both roots cannot be less than θ^{*I} because of the following contradiction.

$$\theta_1^R = p-\frac{\lambda Q+\varepsilon}{2}-\sqrt{\left(p-\frac{\lambda Q+\varepsilon}{2}\right)^2-\Omega\left[\theta^{*I}\right]} < \theta^{*I} \Longrightarrow 2\left(p-\frac{\lambda Q+\varepsilon}{2}\right)\theta^{*I}-(\theta^{*I})^2 > \Omega\left[\theta^{*I}\right]$$

$$\theta_2^R = p-\frac{\lambda Q+\varepsilon}{2}+\sqrt{\left(p-\frac{\lambda Q+\varepsilon}{2}\right)^2-\Omega\left[\theta^{*I}\right]} < \theta^{*I} \Longrightarrow 2\left(p-\frac{\lambda Q+\varepsilon}{2}\right)\theta^{*I}-(\theta^{*I})^2 < \Omega\left[\theta^{*I}\right].$$

As one of the roots must be less than θ^{*I} for $\pi(\theta_i,\theta^{*I})$ to be declining at $\theta_i=\theta^{*I}$ and as $\theta_2^R>\theta_1^R$, the root $\theta_1^R<\theta^{*I}$. Thus $\pi(\theta_i,\theta^{*I})$ has only one stationary point in the interval which is a maximum and assumes a value of δ at $\theta_i=\theta^{*I}$. Hence the value of the function is greater than δ over the interval.

4. The profit function $\pi(\theta_i,\theta^{*I})$ is twice differentiable for $\theta^{*I}<\theta_i<\hat{\theta}(\theta^{*I})+\varepsilon<\theta^{*I}+2\varepsilon$. The derivative of the profit function with respect to θ_i is $\frac{1}{4\lambda\varepsilon}$ times the expression,

$$(p-\theta_i)^2-\left[\begin{array}{c}\left(\frac{\theta^{*I}-\varepsilon+\theta_i-\varepsilon}{2}-\theta_i+\frac{2\varepsilon\lambda Q}{\theta^{*I}+\varepsilon-(\theta_i-\varepsilon)}\right)^2\\ +2\left(\int_{\theta_i-\varepsilon}^{\hat{\theta}(\theta^{*I})}\left(\frac{\theta^{*I}+\theta-\varepsilon}{2}-\theta_i+\frac{2\lambda\varepsilon Q}{\theta^{*I}-(\theta-\varepsilon)}\right)d\theta+2\int_{\hat{\theta}(\theta^{*I})}^{\theta_i+\varepsilon}(p-\theta_i)\,d\theta\right)\end{array}\right] \tag{22}$$

where the term within square brackets is positive. In particular, note that

$$\frac{\theta^{*I}-\varepsilon+\theta_i-\varepsilon}{2}-\theta_i+\frac{2\varepsilon\lambda Q}{\theta^{*I}+\varepsilon-(\theta_i-\varepsilon)}>0.$$

The second order cross partial derivative, $\frac{\partial^2 \pi(\theta_i,\theta^{*I})}{\partial\theta^{*I}\partial\theta_i}$, is given by $\frac{1}{4\lambda\varepsilon}$ times the expression,

$$-\left[\begin{array}{c}\left(\frac{\theta^{*I}-\varepsilon+\theta_i-\varepsilon}{2}-\theta_i+\frac{2\varepsilon\lambda Q}{\theta^{*I}+\varepsilon-(\theta_i-\varepsilon)}\right)\left(1-\frac{4\varepsilon\lambda Q}{(\theta^{*I}+2\varepsilon-\theta_i)^2}\right)\\ \int_{\theta_i-\varepsilon}^{\hat{\theta}\left(\theta^{*I}\right)}\left(1-\frac{4\varepsilon\lambda Q}{(\theta^{*I}+\varepsilon-\theta)^2}\right)d\theta\end{array}\right]$$

By Assumption 1, $4\varepsilon\lambda Q > 4\varepsilon^2 \geq \left(\theta^{*I}+2\varepsilon-\theta_i\right)^2$ which implies that both terms within the square brackets is negative. Hence $\frac{\partial^2 \pi(\theta_i,\theta^{*I})}{\partial\theta^{*I}\partial\theta_i} \geq 0$ for this region and the function $\pi(\theta_i,\theta^{*I})$ has the single crossing property.

Together with the fact that as $\theta_i \longrightarrow \theta^{*I}$, the slope of the function $\pi(\theta_i,\theta^{*I})$ converges to the slope of the function $\pi(\theta^{*I})$ which is negative for any given θ^{*I}, the single crossing property implies that the slope of $\pi(\theta_i,\theta^{*I})$ cannot be positive for higher values of θ_i for which $\theta^{*I} < \theta_i$. Hence, the required inequality is satisfied.

5. For, $\hat{\theta}(\theta^{*I})+\varepsilon < \theta_i < \theta^{*I}+2\varepsilon$, as the profit function equals

$$\pi(\theta_i,\theta^{*I}) = \frac{1}{4\lambda\varepsilon}\int_{\theta_i-\varepsilon}^{\theta_i+\varepsilon}(p-\theta_i)^2\,d\theta = \frac{1}{4\lambda\varepsilon}(p-\theta_i)^2.$$

The derivative is given by

$$-\frac{1}{2\lambda}(p-\theta_i),$$

which is negative for admissible values of p. Hence the condition is satisfied and this proves the proposition.

Finally, to see why $\theta^{*I} < p-\sqrt{2\delta\lambda}$, note that following equation (18)

$$\frac{\partial\pi\left(\theta^{*I},\theta^{*I}\right)}{\partial\theta^{*I}} < 0$$

Since $\pi\left(\theta^{*I},\theta^{*I}\right)=\delta$, we have

$$\frac{d\theta^{*I}}{d\delta} = \frac{1}{\frac{\partial\pi\left(\theta^{*I},\theta^{*I}\right)}{\partial\theta^{*I}}} < 0.$$

We have already shown that $\theta^{*I} = p-\sqrt{2\delta\lambda}$ for $\delta = \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$. Therefore, $\delta > \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2 \Rightarrow \theta^{*I} < p-\sqrt{2\delta\lambda}$.

8.4 Mass of entrants under full and incomplete information

We begin by noting that $\hat{\theta} = p+\varepsilon-\sqrt{2\lambda\delta+4\varepsilon\lambda Q} \leq \theta^{*I}+\varepsilon$ because $(p-\theta^{*I})^2 \leq (\lambda Q+\varepsilon)^2 < 2\lambda\delta+4\varepsilon\lambda Q$ for all $\delta > \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$. Note in particular that for $\delta = \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2$, $\theta^{*I} = p-\sqrt{2\delta\lambda}$, and $\hat{\theta} = \hat{\theta}\left(\theta^{*I}\right) = \theta^{*I}-\varepsilon = p-\lambda Q = p-\sqrt{2\delta\lambda}-\varepsilon$.

Figure 7 depicts the function $\alpha^I-\alpha^F$. The functional form is as follows:

A. For $\theta^l \leq \theta \leq \theta^{*I} - \varepsilon$, as $\alpha^I = 1$,

$$\alpha^I - \alpha^F = 1 - \frac{\sqrt{2\delta\lambda + 4\varepsilon\lambda Q} - \sqrt{2\delta\lambda}}{2\varepsilon} \tag{23}$$

which is constant in θ.

B. For $\theta^{*I} - \varepsilon \leq \theta \leq \hat{\theta} \leq \theta^{*I} + \varepsilon$,

$$\alpha^I - \alpha^F = \frac{\theta^{*I} - (\theta - \varepsilon)}{2\varepsilon} - \frac{\sqrt{2\delta\lambda + 4\varepsilon\lambda Q} - \sqrt{2\delta\lambda}}{2\varepsilon} \tag{24}$$

which is linear and decreasing in θ.

C. For $\hat{\theta} \leq \theta \leq \theta^{*I} + \varepsilon$,

$$\alpha^I - \alpha^F = \frac{\theta^{*I} - (p - \sqrt{2\lambda\delta})}{2\varepsilon} \tag{25}$$

after due simplification and is a negative constant in θ.

D. For $\theta^{*I} + \varepsilon \leq \theta \leq p - \sqrt{2\lambda\delta} + \varepsilon$, as $\alpha^I = 0$,

$$\alpha^I - \alpha^F = \frac{(\theta - \varepsilon) - (p - \sqrt{2\lambda\delta})}{2\varepsilon} \tag{26}$$

after due simplification and is linear and increasing in θ.

E. For $\theta > p - \sqrt{2\lambda\delta} + \varepsilon$, the active mass is zero under both incomplete and full information. Hence,$(\alpha^I - \alpha^F) = 0$

8.5 Appendix: Response of $\alpha^I - \alpha^F$ to changes in Q and δ

8.5.1 Changes in equilibrium thresholds

Recall that the conditional payoff function of a firm with cost parameter θ_i, $\pi(\theta \mid \theta_i, \theta^{*I})$, has the form,

$$\begin{aligned}
\pi(\theta \mid \theta_i, \theta^{*I}) &= \frac{1}{2\lambda}\Big(\frac{(\theta^{*I} + \theta - \epsilon}{2} + \frac{2\epsilon\lambda Q}{\theta^{*I} - \theta + \epsilon} - \theta_i\Big)^2, \text{for, } \theta \leq \hat{\theta}(\theta^{*I}) \\
&= \frac{1}{2\lambda}\Big(p - \theta_i\Big)^2, \text{for, } \theta > \hat{\theta}(\theta^{*I})
\end{aligned}$$

where the cutoff $\hat{\theta}(\theta^{*I}) = (p + \epsilon) - \sqrt{(p - \theta^{*I})^2 + 4\epsilon\lambda Q}$ and $E\Big(\pi(\theta \mid \theta^{*I}, \theta^{*I})\Big) = \delta$ for the pivotal firm.

Consider an increase in Q from an initial level of Q_0 to $Q_1 > Q_0$ and all the other parameters held constant at their initial levels. For the given θ^{*I}, the value of the strictly convex portion of the conditional payoff function increases at any θ and the cutoff $\hat{\theta}(\theta^{*I})$ decreases, as a result of this change. Thus, an

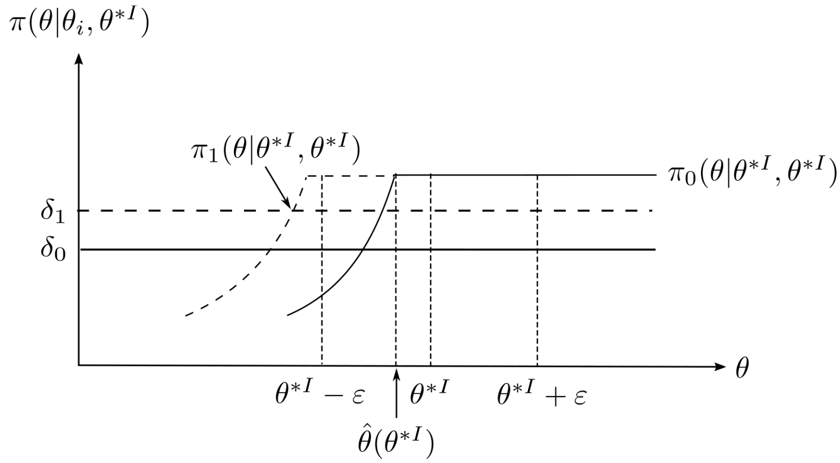


Figure 9: Parametric shifts under incomplete information.

increase in Q results in a shift of the conditional payoff function of the firm with cost parameter $\theta_i = \theta^{*I}$. This is shown in figure 9 where $\pi_0(\theta \mid \theta^{*I}, \theta^{*I})$ corresponds to the initial values of all parameters and $\pi_1(\theta \mid \theta^{*I}, \theta^{*I})$ corresponds to Q_1, all others constant. Hence, the expected payoff of this firm over $\theta \in [\theta^{*I} - \epsilon, \theta^{*I} + \epsilon]$ is strictly greater than δ and the firm with $\theta_i = \theta^{*I}$ is no longer pivotal. The conditional payoff function of the new pivotal firm must be further to the right of $\pi_0(.)$ with a flat stretch that is lower than that of $\pi_0(.)$. The new equilibrium threshold must therefore be greater than θ^{*I}.

Next consider an increase in δ from an initial level of δ_0 to $\delta_1 > \delta_0$ and all the other parameters held constant at their initial levels. For the given θ^{*I} we have as a result, $E\Big(\pi(\theta \mid \theta^{*I}, \theta^{*I})\Big) < \delta_1$ and the firm with $\theta_i = \theta^{*I}$ is no longer pivotal. The conditional payoff function of the new pivotal firm must be to the left of $\pi_0(.)$ in figure 9 with a flat stretch higher than that of $\pi_0(.)$. The new equilibrium threshold therefore must be smaller than θ^{*I}.

The equilibrium threshold function under full information, θ^{*F}, moves in a similar direction as θ^{*I} in response to changes in these parameters. Recall from proposition 2 that

$$\begin{aligned}\theta^{*F} &= \theta - \epsilon + \sqrt{2\lambda\delta + 4\epsilon\lambda Q} - \sqrt{2\lambda\delta}, \text{for, } \theta \leq \hat{\theta} \\ &= p - \sqrt{2\lambda\delta}, \text{for, } \theta > \hat{\theta}\end{aligned}$$

where $\hat{\theta} = p + \epsilon - \sqrt{2\lambda\delta + 4\epsilon\lambda Q}$.

An increase in Q shifts the strictly increasing part of θ^{*F} up and the cutoff point $\hat{\theta}$ to the left. This is shown in figure 10 where θ_0^{*F} corresponds to the initial values of the parameters and θ_1^{*F} corresponds to Q_1, all else constant. Thus, an increase in Q increases the full information equilibrium threshold for a given $\theta \leq \hat{\theta}$. An increase in δ shifts the point $\hat{\theta}$ left and both parts of θ^{*F} down as shown by the curve θ_2^{*F} in the figure. Thus, the equilibrium full information threshold is lowered for a given θ as a result of this change.

Under both scenarios, an increase in industry capacity, Q, provides room for more firms to enter whereas

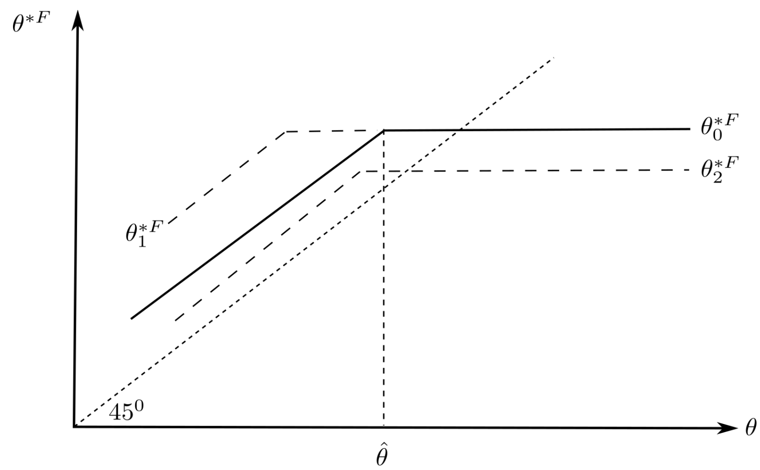


Figure 10: Parametric shifts under full information.

an increase in the capital cost, δ, restricts entry potentially. However, as the equilibrium entry thresholds move in the same direction, it is difficult to predict how the excess entry function $\alpha^I - \alpha^F$ behaves for each θ. In fact, the cardinal point $\tilde{\theta}$ in figure 8 can shift either right or left depending on the values of the other parameters. It is nevertheless possible to say how the function behaves in the regions A and C.

8.5.2 Changes in mass of entrants

Recall that under full information, for the parametric configuration we are interested in (namely, $\delta > \frac{1}{2\lambda}(\lambda Q - \epsilon)^2$), only a proper subset of the firms enter the industry in any equilibrium. Call this partial entry. Under incomplete information on the other hand, the region A is characterized by the feature that all firms enter.

A rise in Q increases the length of the region A because of an increase in θ^{*I}. The value of $\alpha^I - \alpha^F$, a positive constant, diminishes over this region because of a rise in α^F. Thus, the extent of excess entry due to placement uncertainty is lower but this phenomenon is prevalent over a wider range of θ values as the region A is wider.

An increase in δ on the other hand reduces the length of the region A because of a decrease in θ^{*I}, while increasing the value of $\alpha^I - \alpha^F$ because of a fall in α^F. Thus, an increase in capital costs increases excess entry due to placement uncertainty but over a smaller range of θ values as the region A is narrower.

An increase in Q increases the length of the region C as $\hat{\theta}$ shifts left and $\theta^{*I} + \epsilon$ shifts right. The value of $\alpha^I - \alpha^F$ increases but stays negative, as θ^{*I} increases without α^F changing. Thus the extent of under entry because of placement uncertainty is lower but this happens over a wider region C.

The effect of a rise in δ on the length of the region C is indeterminate as both $\hat{\theta}$ and $\theta^{*I} + \epsilon$ move in the same direction, leftwards. The value of $\alpha^I - \alpha^F$ can also be affected either way.

9 Extended Appendix: Placement bias

This section considers the effect of an over-placement bias on a firm's part on its entry decision and the resulting industry performance. We continue to assume that firms have the common belief that $\theta_i \sim U[\theta - \varepsilon, \theta + \varepsilon]$ where θ is unobserved. Now, however, a firm believes that it has a better than average cost efficiency. Thus, when entry decisions are made, a firm with cost parameter θ_i believes that the unobserved θ follows the distribution, $(\theta \mid \theta_i) \sim U[\theta_i + \beta - \varepsilon, \theta_i + \beta + \varepsilon]$ where $\beta \leq \varepsilon$ represents the over-placement bias. Specifically, this implies that from the firm's point of view, $E[\theta \mid \theta_i] = \theta_i + \beta > \theta_i$.[18] The question of whether such a bias persists with learning is outside the purview of our one shot game framework and left as a future exercise.

Denote by $\theta^{*I,\beta}$ the common entry threshold. Then, equilibrium $\theta^{*I,\beta}$ is characterized by,

$$\frac{1}{2\varepsilon} \int_{\theta^{*I,\beta}+\beta-\varepsilon}^{\theta^{*I,\beta}+\beta+\varepsilon} \pi(\theta^{*I,\beta}, \theta^{*I,\beta}) d\theta = \delta.$$

where, the form of key expressions are the same as before. The equilibrium threshold is now a function of all the model parameters including β.

9.1 Equilibrium with placement bias

Proposition 6. *1. For $\delta \in \left[0, \frac{1}{2\lambda}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)^2\right]$, a unique pure strategy Bayesian Nash equilibrium in switching strategies exist. The equilibrium threshold is the solution to the equation:*

$$\frac{1}{4\lambda\varepsilon} \int_{\theta^{*I,\beta}+\beta-\varepsilon}^{\theta^{*I,\beta}+\beta+\varepsilon} (p - \theta^{*I,\beta})^2 = \delta \tag{27}$$

Specifically, $\theta^{*I,\beta} = p - \sqrt{2\lambda\delta}$.

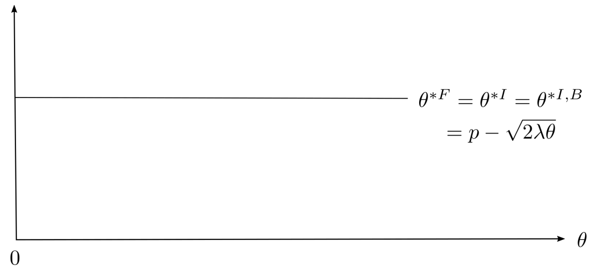
2. For $\delta \in (\frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2})^2, \frac{(\lambda Q+\beta)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda}]$, a unique pure strategy Bayesian Nash equilibrium in switching strategies exist. The equilibrium threshold solves the equation,

$$\frac{1}{4\lambda\varepsilon}[\int_{\theta^{*I,\beta}+\beta-\varepsilon}^{\hat{\theta}(\theta^{*I,\beta})} (\frac{\theta^{*I,\beta}+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{\theta^{*I,\beta}-(\theta-\varepsilon)} - \theta^{*I,\beta})^2 d\theta + \int_{\hat{\theta}(\theta^{*I,\beta})}^{\theta^{*I,\beta}+\beta+\varepsilon} (p - \theta^{*I,\beta})^2 d\theta)] = \delta \tag{28}$$

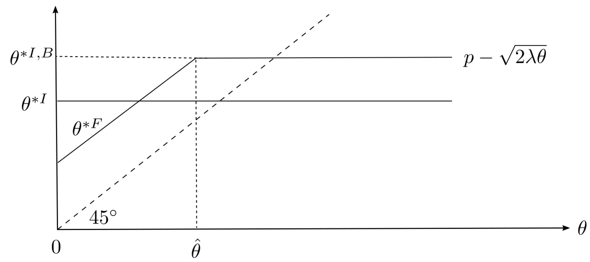
where $\hat{\theta}(\theta^{*I,\beta}) = (p+\varepsilon) - \sqrt{(p-\theta^{*I,\beta})^2 + 4\varepsilon\lambda Q}$.

3. $\theta^{*I,\beta} = \theta^{*I} = \theta^{*F}$ for $\delta \in \left[0, \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2\right]$ whereas $\theta^{*I,\beta} > \theta^{*I}$ for $\delta \in \left(\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2, \frac{(\lambda Q)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda}\right]$.

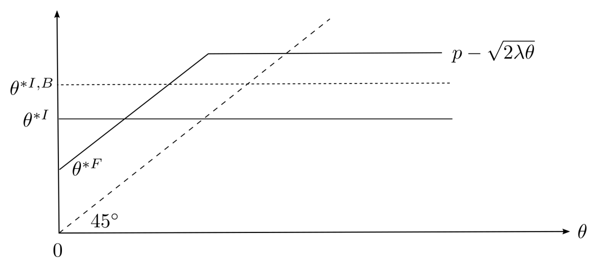
[18] $\beta \leq \varepsilon$ ensures that $E[\theta \mid \theta_i] = \theta_i + \beta \leq \theta_i + \varepsilon$ and consistency with the common belief assumption.



(a) $\delta \in \left(0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2\right)$



(b) $\delta \in \left(\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2, \frac{1}{2\lambda}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)^2\right)$



(c) $\delta \in \left(\frac{1}{2\lambda}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)^2, \left(\frac{(\lambda Q)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda}\right)\right)$

Figure 11: **Entry thresholds and capital costs.**

PROOF: SEE APPENDIX 9.3.

Proposition 6 divide up the parameter space into three regions over which all the three thresholds, θ^{*F}, θ^{*I} and $\theta^{*I,\beta}$, are defined.

REGION I: All three thresholds are identical and equal to $p - \sqrt{2\lambda\delta}$ for $\delta \in \left[0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2\right]$ and are independent of θ. Thus placement bias does not matter. Figure 11a depicts this case.

REGION II: For $\delta \in \left(\frac{1}{2\lambda}\left(\lambda Q - \varepsilon\right)^2, \frac{1}{2\lambda}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)^2\right]$ the threshold with bias, $\theta^{*I,\beta} = p - \sqrt{2\lambda\delta}$ is independent of θ and higher than both θ^{*F} and θ^{*I}. The relationship between θ^{*F} and θ^{*I} is as shown in figure 6. The relationship between all three is as depicted in figure 11b.

REGION III: For $\delta \in \left(\frac{1}{2\lambda}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)^2, \frac{(\lambda Q)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda}\right]$ the threshold, $\theta^{*I,\beta} < p - \sqrt{2\lambda Q}$ but greater than θ^{*I}. Since θ^{*F} is increasing in θ and attains a ceiling of $p - \sqrt{2\lambda Q}$, it follows that θ^{*F} is less than $\theta^{*I,\beta}$ for lower values of θ and higher than $\theta^{*I,\beta}$ for higher values of θ. The relationships are depicted in figure

11c.

9.2 Mass of entrants

Denote by, $\alpha^{I,\beta}$, the mass of entrants under uncertainty with a placement bias.

For parametric configurations in Region I, for all values of θ, $\alpha^{I,\beta} = \alpha^I = \alpha^F$ as the three thresholds are identical. Thus, placement uncertainty with or without bias does not cause excessive entry for low capital costs.

For parametric configurations in regions II and III, note that $\alpha^{I,\beta} - \alpha^F = (\alpha^{I,\beta} - \alpha^I) + (\alpha^I - \alpha^F)$, where $\alpha^{I,\beta} - \alpha^I \geq 0$ as $\theta^{*I,\beta} > \theta^{*I}$. The sign of $(\alpha^I - \alpha^F)$ is conditional as in section 6.

So far as Region II is concerned however, $\alpha^{I,\beta} \geq \alpha^F$ as $\theta^{*I,\beta} > \theta^{*F}$ irrespective of the sign of $(\alpha^I - \alpha^F)$. Thus, placement uncertainty causes excess entry for parameters lying in this region, at least part of which may be attributed to over-placement bias. While acknowledging an identification problem here, it is important to distinguish between the two different sources of excess entry - pure uncertainty and over-placement bias.

In Region III, as $\alpha^{I,\beta} - \alpha^I \geq 0$, over-placement bias exacerbates excess entry for low values of θ and mitigates under entry for high values of θ relative to the full information benchmark. We summarise all these conclusions in the following Proposition.

Proposition 7. *I. When $\delta \in [0, \frac{1}{2\lambda}(\lambda Q - \varepsilon)^2]$, placement uncertainty does not lead to excess entry.*

II. When $\delta \in (\frac{1}{2\lambda}(\lambda Q - \varepsilon)^2, \frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2})^2]$, placement uncertainty causes excess entry, part of which at least can be attributed to over-placement bias.

III. When $\delta \in [\frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2})^2, \frac{(\lambda Q)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda}]$, over-placement bias reinforces a positive $(\alpha^I - \alpha^F)$ effect and mitigates a negative $(\alpha^I - \alpha^F)$ effect.

9.3 Proof of Proposition 6

Part I:

Key expressions of the model have the same functional form, with or without bias. Hence, $r(\theta, \theta^{*I,\beta}) = 0$ for all $\theta \in [\theta^{*I,\beta} + \beta - \varepsilon, \theta^{*I,\beta} + \beta + \varepsilon]$, iff

$$r(\theta^{*I,\beta} + \beta - \varepsilon, \theta^{*I,\beta}) = 0$$

which on substitution is equivalent to the condition, $p - \frac{2\theta^{*I,\beta}+\beta-2\varepsilon}{2} - \frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} \leq 0$. This simplifies to $p - \theta^{*I,\beta} \leq \frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}$.

When $r(\theta, \theta^{*I,\beta}) = 0$ for all $\theta \in [\theta^{*I,\beta} + \beta - \varepsilon, \theta^{*I,\beta} + \beta + \varepsilon]$, the equilibrium $\theta^{*I,\beta}$ is given by the solution of

$$\frac{1}{2\varepsilon}\int_{\theta^{*I,\beta}+\beta-\varepsilon}^{\theta^{*I,\beta}+\beta+\varepsilon} \frac{1}{2\lambda}(p - \theta^{*I,\beta})^2 d\theta = \delta$$

or $\theta^{*I,\beta} = p - \sqrt{2\lambda\delta}$.

Substituting the expression for $\theta^{*I,\beta}$ in the previous expression, we obtain the required range restriction on the parameters for the type (1) equilibrium.

$$\delta \leq \frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2})^2$$

Next, we need to show that the solution $\theta^{*I,\beta}=p-\sqrt{2\lambda\delta}$ is an equilibrium.

As we are looking for an equilibrium under which $r(\theta,\theta^{*I,\beta})=0$ for all $\theta\in[\theta^{*I,\beta}+\beta-\varepsilon,\theta^{*I,\beta}+\beta+\varepsilon]$, at the value of θ at which the equilibrium permit price falls to zero, all firms enter. Hence the value of θ at which the equilibrium permit price equals zero is given by $\hat{\theta}=p-\lambda Q<\theta^{*I,\beta}+\beta-\varepsilon$.

The individual profit function takes on three different forms depending on three zones in which θ_i may lie. We discuss the condition that each must satisfy and show that these are met.

1. For $\theta_i<\theta^{*I,\beta}-\beta-2\varepsilon$, we require

$$\pi(\theta_i,\theta^{*I,\beta})=\frac{1}{4\lambda\varepsilon}\int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon}(\theta-\theta_i+\lambda Q)^2\,d\theta>\delta$$

   The condition is true for $\beta=0$. Since the integrand is increasing and convex in θ, the above is true for $\beta>0$ as well.

2. For $\theta^{*I,\beta}-\beta-2\varepsilon\leq\theta_i\leq\theta^{*I,\beta}-\beta$, the profit function and the required condition are

$$\pi(\theta_i,\theta^{*I,\beta})=\frac{1}{4\lambda\varepsilon}\int_{\theta_i+\beta-\varepsilon}^{p-\lambda Q}(\theta-\theta_i+\lambda Q)^2\,d\theta+\frac{1}{4\lambda\varepsilon}\int_{p-\lambda Q}^{\theta_i+\beta+\varepsilon}(p-\theta_i)^2\,d\theta>\delta$$

   Upon simplification,

$$\pi(\theta_i,\theta^{*I,\beta})=\frac{1}{4\lambda\varepsilon}\left[\frac{(p-\theta_i)^3}{3}-\frac{(\lambda Q+\beta-\varepsilon)^3}{3}+(p-\theta_i)^2(\theta_i+\beta+\varepsilon-p+\lambda Q)\right]$$

   The expression equals $\frac{(\lambda Q+\beta)^2}{2\lambda}+\frac{\varepsilon^2}{6\lambda}$ when $\theta_i+\beta+\varepsilon=p-\lambda Q$, implying that the profit function is continuous at this value of θ_i.

   It is easy to show that the derivative of the function with respect to θ_i is negative. Hence, $\pi(\theta_i,\theta^{*I,\beta})$ is strictly declining in θ_i through the range under consideration. Since $\pi(\theta_i,\theta^{*I,\beta})\longrightarrow\delta$ as $\theta_i\longrightarrow\theta^{*I,\beta}$, the condition is satisfied.

3. When $\theta^{*I,\beta}-\beta<\theta_i<\theta^{*I,\beta}$, we must have

$$\pi(\theta_i,\theta^{*I,\beta})=\frac{1}{2\varepsilon}\int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon}\frac{1}{2\lambda}(p-\theta_i)^2\geq\delta$$

Since $\frac{1}{2\varepsilon}\int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon}\frac{1}{2\lambda}\left(p-\theta_i\right)^2=\frac{1}{2\lambda}\left(p-\theta_i\right)^2$, and the latter is strictly declining in θ_i, the condition is satisfied, because at $\theta_i=\theta^{*I,\beta}$, $\frac{1}{2\lambda}\left(p-\theta_i\right)^2=\delta$.

4. When $\theta^{*I,\beta}<\theta_i$, we must have

$$\pi(\theta_i,\mu^*)=\frac{1}{2\varepsilon}\int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon}\frac{1}{2\lambda}\left(p-\theta_i\right)^2<\delta$$

By the same arguments as in the previous step, the condition is satisfied.

Part 2:

As in the case of $\beta=0$, the Proposition will be proved through multiple steps.

STEP 1.: To show that the interval $(\frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2})^2,\frac{(\lambda Q+\beta)^2}{2\lambda}+\frac{\varepsilon^2}{6\lambda}]$ is non-empty, it suffices to show that under appropriate restrictions on the parameters,

$$\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2}\right)^2<(\lambda Q+\beta)^2,\text{ or, }\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2}\right)<(\lambda Q+\beta)$$

which on simplification turns out to be

$$\lambda Q<\frac{(2\varepsilon-\beta)(2\varepsilon+\beta)}{2\beta}$$

For any value of $\beta\in(0,\varepsilon]$, there exists an upper bound on λQ, for which the above inequality is satisfied. When $\beta=\varepsilon$, the desired interval is non-empty if $\lambda Q<\frac{3}{2}\varepsilon$. The upper bound is higher for lower β.

STEP II: We next show that there is a unique solution $\theta^{*I,\beta}$ that satisfy (28).

Consider the function,

$$\pi(k)=\frac{1}{2\varepsilon}\int_{k+\beta-\varepsilon}^{\hat{\theta}(k)}\frac{1}{2\lambda}\left(\frac{k+\theta-\varepsilon}{2}+\frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}-k\right)^2d\theta+\frac{1}{2\varepsilon}\int_{\hat{\theta}(k)}^{k+\beta+\varepsilon}\frac{1}{2\lambda}(p-k)^2d\theta$$

where $\hat{\theta}(k)=p+\varepsilon-\sqrt{\left(p-k\right)^2+4\varepsilon\lambda Q}$. On simplification,

$$\begin{aligned}\pi(k)=&\frac{1}{4\lambda\varepsilon}\int_{k+\beta-\varepsilon}^{\hat{\theta}(k)}\left(\frac{4\left(\varepsilon\lambda Q\right)^2}{\left(\theta-(k+\varepsilon)\right)^2}+\frac{\left(\theta-(k+\varepsilon)\right)^2}{4}-2\varepsilon\lambda Q\right)d\theta\\&+\frac{1}{4\lambda\varepsilon}\left(p-k\right)^2\left[(k+\beta-p)+\sqrt{\left(p-k\right)^2+4\varepsilon\lambda Q}\right]\end{aligned}$$

As before, a change of variable $z=\theta-(k+\varepsilon)$ allows us to evaluate the first integral. With this change of variable, the upper and lower limits of the integration are, respectively, $\hat{\theta}\left(k\right)-k-\varepsilon=(p-k)-\sqrt{\left(p-k\right)^2+4\varepsilon\lambda Q}$ and $\beta-2\varepsilon$. Evaluating the integral using the new variable and then substituting the new variable back and simplifying, we have,

$$\pi(k) = \frac{1}{4\lambda\varepsilon}\left[\begin{array}{c} -\frac{4(\varepsilon\lambda Q)^2}{(p-k)-\sqrt{(p-k)^2+4\varepsilon\lambda Q}} - \frac{4(\varepsilon\lambda Q)^2}{2\varepsilon} + \frac{\left((p-k)-\sqrt{(p-k)^2+4\varepsilon\lambda Q}\right)^3}{12} \\ +\frac{(2\varepsilon-\beta)^3}{12} - 2\varepsilon\lambda Q\left[(p-k)-\sqrt{(p-k)^2+4\varepsilon\lambda Q}+2\varepsilon-\beta\right] \\ -\left[(p-k)-\beta-\sqrt{(p-k)^2+4\varepsilon\lambda Q}\right](p-k)^2 \end{array}\right] \tag{29}$$

We try to show next that $\frac{d\pi(k)}{dk} < 0$. A second change of variable helps us to do that. Define

$$x \equiv \sqrt{(p-k)^2 + 4\varepsilon\lambda Q} - (p-k) > 0$$

and note that

$$\frac{dx}{dk} = 1 - \frac{p-k}{\sqrt{(p-k)^2+4\varepsilon\lambda Q}} > 0$$

Further note that $(p-k)^2 = \frac{x^2}{4} + \frac{(2\varepsilon\lambda Q)^2}{x^2} - 2\varepsilon\lambda Q$.

With the second change of variable, $\pi(k)$ can be rewritten as

$$\pi(k) = \frac{1}{4\lambda\varepsilon}\left[\begin{array}{c} -\frac{4(\varepsilon\lambda Q)^2}{2\varepsilon-\beta} + \frac{(2\varepsilon-\beta)^3}{12} - 2\varepsilon\lambda Q(2\varepsilon-\beta) \\ +\frac{x^3}{6} + \frac{8(\varepsilon\lambda Q)^2}{x} + \beta(\frac{x}{2} - \frac{2\varepsilon\lambda Q}{x})^2 \end{array}\right]$$

Thus, whether $\pi(k)$ is increasing or decreasing in k depends on whether it decreases or increases in x.

$$\begin{aligned}\frac{d\pi}{dx} &= \frac{1}{4\lambda\varepsilon}\left[\frac{x^2}{2} - \frac{8(\varepsilon\lambda Q)^2}{x^2}\right] + \beta\left(\frac{x}{2} - \frac{2\varepsilon\lambda Q}{x}\right)\left(\frac{1}{2} + \frac{2\varepsilon\lambda Q}{x}\right) \\ &= \frac{1}{4\lambda\varepsilon}\left(\frac{x}{\sqrt{2}} + \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right)\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right) + \beta\left(\frac{x}{2} - \frac{2\varepsilon\lambda Q}{x}\right)\left(\frac{1}{2} + \frac{2\varepsilon\lambda Q}{x}\right)\end{aligned}$$

Thus the sign of $\frac{d\pi(k)}{dk}$ depends on the signs of the terms, $\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right)$ and $\left(\frac{x}{2} - \frac{2\varepsilon\lambda Q}{x}\right)$.

Substituting the expression for x back and simplifying, it is straightforward to check that so long as $(p-k) > 0$ (true for values of k we are interested in),

$$\left(\frac{x}{\sqrt{2}} - \frac{2\sqrt{2}\varepsilon\lambda Q}{x}\right) = -\sqrt{2}(p-k) < 0$$

and

$$\left(\frac{x}{2} - \frac{2\varepsilon\lambda Q}{x}\right) = -(p-k) < 0$$

Thus $\frac{d\pi(k)}{dk} < 0$.

We next need to show that the function $\pi(k)$ is greater than δ for some k and less than δ for some k.

As before, for any given k, the following inequality is true.

$$\pi(k) \leq \frac{1}{2\varepsilon}\int_{k+\beta-\varepsilon}^{k+\beta+\varepsilon} \frac{1}{2\lambda}\left(p-k\right)^2 d\theta = \frac{1}{2\lambda}(p-k)^2 \tag{30}$$

Moreover, for $k = p - (\lambda Q - \varepsilon)$, $\pi(k) \leq \frac{1}{2\lambda}(p-k)^2 = \frac{1}{2\lambda}\left(\lambda Q - \varepsilon\right)^2$. Since, $\frac{1}{2\lambda}\left(\lambda Q - \varepsilon\right)^2 < (\frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2})^2 < \delta$ for $0 < \beta \leq 2\varepsilon$, $\pi(k) < \delta$ for some value of k.

Similarly, for any given k, the following inequality is true for $\theta \leq p - \lambda Q$.

$$\frac{1}{2\varepsilon}\int_{k+\beta-\varepsilon}^{k+\beta+\varepsilon} \frac{1}{2\lambda}\left(\theta + \lambda Q - k\right)^2 d\theta \leq \pi(k) \tag{31}$$

The inequality is true by the following arguments. At $\theta = k + \beta - \varepsilon$, the expressions within integral signs have following values:

$$\theta + \lambda Q - k = \lambda Q + \beta - \varepsilon$$

$$\left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)} - k\right) = \left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right)$$

Note that,

$$\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - \frac{2\varepsilon-\beta}{2}\right) - (\lambda Q + \beta - \varepsilon) = \frac{\beta}{2}\left(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} - 1\right) > 0,$$

since $\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta} > 1$.

Hence, for $\theta = k + \beta - \varepsilon$,

$$\theta + \lambda Q - k < \left(\frac{k+\theta-\varepsilon}{2} + \frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}\right)$$

Both functions are increasing in θ, but the slope of $\theta + \lambda Q$ is 1 and following the same steps as in proposition 3, we can show that the slope of the RHS expression is greater than 1.

Hence, for any given k and $\theta \leq p - \lambda Q$, inequality (31) is true.

Thus, for $k = p - \lambda Q - \varepsilon$,

$$\pi(k) \geq \frac{1}{2\varepsilon}\int_{k-\varepsilon}^{k+\varepsilon} \frac{1}{2\lambda}\left(\theta + \lambda Q - k\right)^2 d\theta = \frac{1}{12\lambda\varepsilon}\left[(\lambda Q + \beta + \varepsilon)^3 - (\lambda Q + \beta - \varepsilon)^3\right] = \frac{(\lambda Q + \beta)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda} \geq \delta$$

Hence $\pi(k)$ has a unique intersection $k = \theta^{*I,\beta}$ with δ.

Step III: We next show that the switching strategy with the threshold μ^* is an equilibrium. As before, we need to show that for any firm of type θ_i, $\pi(\theta_i, \theta^{*I,\beta}) > \delta$ for $\theta_i < \theta^{*I,\beta}$ and $\pi(\theta_i, \theta^{*I,\beta}) < \delta$ for $\theta_i > \theta^{*I,\beta}$.

The following list characterizes the individual profit function $\pi(\theta_i, \theta^{*I,\beta})$, for each zone in which θ_i may lie and provides the condition that the profit function must satisfy. The rational for the form of the profit function for each zone is identical to that for the no-bias ($\beta = 0$) case and is therefore omitted. The only exception is zone 4 below which is new to $\beta > 0$.

1. $\theta_i < \theta^{*I,\beta} - \beta - 2\varepsilon$

$$\pi(\theta_i, \theta^{*I,\beta}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon} (\theta + \lambda Q - \theta_i)^2 \, d\theta > \delta$$

2. $\theta^{*I,\beta} - \beta - 2\varepsilon < \theta_i < \hat{\theta}(\theta^{*I,\beta}) - \beta - \varepsilon < \theta^{*I,\beta} - \beta$.

$$\begin{aligned}\pi(\theta_i, \theta^{*I,\beta}) = \frac{1}{4\lambda\varepsilon} \Bigg[& \int_{\theta_i+\beta-\varepsilon}^{\theta^{*I,\beta}-\varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta \\ & + \int_{\theta^{*I,\beta}-\varepsilon}^{\theta_i+\beta+\varepsilon} \left(\frac{\theta^{*I,\beta} + \theta - \varepsilon}{2} - \theta_i + \frac{2\varepsilon\lambda Q}{\theta^{*I,\beta} - (\theta - \varepsilon)} \right)^2 d\theta \Bigg] > \delta\end{aligned}$$

3. $\hat{\theta}(\theta^{*I,\beta}) - \beta - \varepsilon < \theta_i < \theta^{*I,\beta} - \beta$

$$\begin{aligned}\pi(\theta_i, \theta^{*I,\beta}) = \frac{1}{4\lambda\varepsilon} \Bigg[& \int_{\theta_i+\beta-\varepsilon}^{\theta^{*I,\beta}-\varepsilon} (\theta - \theta_i + \lambda Q)^2 \, d\theta \\ & + \int_{\theta^{*I,\beta}-\varepsilon}^{\hat{\theta}(\theta^{*I,\beta})} \left(\frac{\theta^{*I,\beta} + \theta - \varepsilon}{2} - \theta_i + \frac{2\varepsilon\lambda Q}{\theta^{*I,\beta} - (\theta - \varepsilon)} \right)^2 d\theta \\ & + \int_{\hat{\theta}(\theta^{*I,\beta})}^{\theta_i+\beta+\varepsilon} (p - \theta_i)^2 \, d\theta \Bigg] > \delta\end{aligned}$$

4. $\theta^{*I,\beta} - \beta < \theta_i < \theta^{*I,\beta}$.

$$\pi(\theta_i, \theta^{*I,\beta}) = \int_{\theta_i+\beta-\varepsilon}^{\hat{\theta}(\theta^{*I,\beta})} \left(\frac{\theta^{*I,\beta} + \theta - \varepsilon}{2} - \theta_i + \frac{2\varepsilon\lambda Q}{\theta^{*I,\beta} - (\theta - \varepsilon)} \right)^2 d\theta + \int_{\hat{\theta}(\theta^{*I,\beta})}^{\theta_i+\beta+\varepsilon} (p - \theta_i)^2 \, d\theta > \delta$$

Since $\theta \in [\theta_i + \beta - \varepsilon, \theta_i + \beta + \varepsilon]$, $\theta^{*I,\beta} - \beta < \theta_i \implies \theta^{*I,\beta} - \varepsilon < \theta$ for all possible values of θ. Hence, it is never the case that all firms are active and this explains the form of the profit function.

5. $\theta^{*I,\beta} < \theta_i < \hat{\theta}(\theta^{*I,\beta}) - \beta + \varepsilon$

$$\pi(\theta_i, \theta^{*I,\beta}) = \int_{\theta_i+\beta-\varepsilon}^{\hat{\theta}(\theta^{*I,\beta})} \left(\frac{\theta^{*I,\beta} + \theta - \varepsilon}{2} - \theta_i + \frac{2\varepsilon\lambda Q}{\theta^{*I,\beta} - (\theta - \varepsilon)} \right)^2 d\theta + \int_{\hat{\theta}(\theta^{*I,\beta})}^{\theta_i+\beta+\varepsilon} (p - \theta_i)^2 \, d\theta < \delta$$

6. $\hat{\theta}(\theta^{*I,\beta}) - \beta + \varepsilon < \theta_i$

$$\pi(\theta_i, \theta^{*I,\beta}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon} (p - \theta_i)^2 \, d\theta < \delta$$

To prove the rest of the proposition, we need to show that the required condition for each zone is satisfied.

1. For $\theta_i < \theta^{*I,\beta} - 2\varepsilon$,

$$\pi(\theta_i, \theta^{*I,\beta}) = \frac{1}{4\lambda\varepsilon} \int_{\theta_i+\beta-\varepsilon}^{\theta_i+\beta+\varepsilon} (\theta + \lambda Q - \theta_i)^2 \, d\theta = \frac{1}{12\lambda\varepsilon} \left[(\lambda Q + \beta + \varepsilon)^3 - (\lambda Q + \beta - \varepsilon)^3 \right]$$

   The last expression equals $\frac{(\lambda Q+\beta)^2}{2\lambda} + \frac{\varepsilon^2}{6\lambda} \geq \delta$.

2. The proof for this region is identical to the one for the case $\beta = 0$ and is hence omitted.

3. We shall verify the inequalities for the next three regions together.

   Using the same arguments as in step III of proposition 3, we note that the slopes of the two functions, $\pi(\theta_i, \theta^{*I,\beta})$ and $\pi(\theta^{*I,\beta})$ must converge as $\theta_i \longrightarrow \theta^{*I,\beta}$ and in particular $\pi(\theta_i, \theta^{*I,\beta})$ must be declining at $\theta_i = \theta^{*I,\beta}$. These statements taken together imply that $\pi(\theta_i, \theta^{*I,\beta})$ must have at least one stationary point that is a maximum in the interval, $[\hat{\theta}(\theta^{*I,\beta}) - \beta - \varepsilon, \theta^{*I,\beta}]$ which includes the region $[\hat{\theta}(\theta^{*I,\beta}) - \beta - \varepsilon, \theta^{*I,\beta} - \beta]$.

   We therefore check the roots of the derivative of $\pi(\theta_i, \theta^{*I,\beta})$ with respect to θ_i.

   The derivatives of the first and the second term of $\pi(\theta_i, \theta^{*I,\beta})$ with respect to θ_i are the same as the derivatives of the first and second terms of $\pi(\theta_i, \theta^{*I})$ in proposition 3.

   The derivative of the third term is given by

$$\frac{1}{4\lambda\varepsilon} \left[3(p - \theta_i)^2 - 2(p - \theta_i) \left(\sqrt{(p - \theta^{*I,\beta})^2 + 4\varepsilon\lambda Q} + \beta \right) \right]$$

   which is more conveniently written as,

$$\frac{1}{4\lambda\varepsilon} \left[3(p - \theta_i)^2 - 2(p - \theta_i) \left(p + \varepsilon - \hat{\theta}(\theta^{*I,\beta}) + \beta \right) \right]$$

   As before, these terms can be combined to get

$$\frac{d\pi(\theta_i, \theta^{*I,\beta})}{d\theta_i} = \frac{1}{2\lambda\varepsilon} \left[\theta_i^2 - 2\left(p - \frac{\lambda Q + \varepsilon + \beta}{2} \right) \theta_i + \Omega\left[\theta^{*I,\beta}, \beta \right] \right] \tag{32}$$

where

$$\Omega\left[\theta^{*I,\beta},\beta\right] \equiv \left(\theta^{*I,\beta}-\varepsilon\right)^2 - \frac{\left(\hat{\theta}(\theta^{*I,\beta})+\mu^*-\varepsilon\right)^2}{4} + 2\varepsilon\lambda Q\log\left[1-\frac{\hat{\theta}(\theta^{*I,\beta})-\left(\theta^{*I,\beta}-\varepsilon\right)}{2\varepsilon}\right]$$
$$+\frac{p^2-\left(\lambda Q+\theta^{*I,\beta}-\varepsilon\right)^2}{2}+p\left(\hat{\theta}(\theta^{*I,\beta})-\varepsilon-\beta\right)$$

Derivative (32) has two roots given by

$$\theta^R_{1,2}\left(\theta^{*I,\beta}\right) \equiv \left(p-\frac{\lambda Q+\varepsilon+\beta}{2}\right) \pm \sqrt{\left(p-\frac{\lambda Q+\varepsilon+\beta}{2}\right)^2-\Omega\left[\theta^{*I,\beta},\beta\right]},$$

Using the same steps as in the case of $\beta = 0$ in proposition 3, we show that both roots cannot be less than $\theta^{*I,\beta}-\beta$ because of a contradiction.

We next show that $\frac{d\pi(\theta_i,\theta^{*I,\beta})}{d\theta_i} < 0$ for $\theta_i \in [\theta^{*I,\beta}-\beta, \hat{\theta}(\theta^{*I,\beta})-\beta+\varepsilon]$. If the last statement is true, then the necessary maxima of $\pi(\theta_i,\theta^{*I,\beta})$ lies in the interval, $[\hat{\theta}(\theta^{*I,\beta})-\beta-\varepsilon, \theta^{*I,\beta}-\beta]$ and is unique.

As the form of the function, $\pi(\theta_i,\theta^{*I,\beta})$, is identical over the sub-intervals $[\theta^{*I,\beta}-\beta,\mu^*]$ and $[\theta^{*I,\beta},\hat{\theta}(\theta^{*I,\beta})-\beta+\varepsilon]$, the arguments put forth for the function, $\pi(\theta_i,\theta^{*I})$ for region 4 in proposition 3 apply and $\frac{d\pi(\theta_i,\theta^{*I,\beta})}{d\theta_i} < 0$ for the entire interval $\theta_i \in [\theta^{*I,\beta}-\beta, \hat{\theta}(\theta^{*I,\beta})-\beta+\varepsilon]$.

Thus $\pi(\theta_i,\theta^{*I,\beta})$ has a unique maxima in $[\hat{\theta}(\theta^{*I,\beta})-\beta-\varepsilon, \theta^{*I,\beta}-\beta]$. Hence all the three required inequalities for regions 3, 4 and 5 are satisfied.

6. The required inequality follows from the same argument provided for region 5 in proposition 3.

Part 3:

The statement is true for $\delta \in [0, \frac{1}{2\lambda}(\lambda Q-\varepsilon)^2]$.

For $\delta \in (\frac{1}{2\lambda}(\lambda Q-\varepsilon)^2, \frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2})^2]$, the threshold $\theta^{*I,\beta} = p-\sqrt{2\delta\lambda}$ whereas the threshold $\theta^{*I} < p-\sqrt{2\delta\lambda}$. Hence statement is true.

For $\delta \in (\frac{1}{2\lambda}(\frac{2\varepsilon\lambda Q}{2\varepsilon-\beta}-\frac{2\varepsilon-\beta}{2})^2, \frac{(\lambda Q)^2}{2\lambda}+\frac{\varepsilon^2}{6\lambda}]$ the entry threshold $\theta^{*I,\beta}$ is determined by the solution of the following equation, for a given β and δ

$$\pi(k,\beta) = \frac{1}{2\varepsilon}\int_{k+\beta-\varepsilon}^{\hat{\theta}(k)}\frac{1}{2\lambda}\left(\frac{k+\theta-\varepsilon}{2}+\frac{2\varepsilon\lambda Q}{k-(\theta-\varepsilon)}-k\right)^2 d\theta + \frac{1}{2\varepsilon}\int_{\hat{\theta}(k)}^{k+\beta+\varepsilon}\frac{1}{2\lambda}(p-k)^2 d\theta = \delta$$

Note that the expression within brackets in the first term is increasing and convex in θ. Hence, as the limits of the integrals are increasing in β, $\pi(k,\beta)$ is increasing in β. Hence, the solution is increasing in β. Δ